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Null controllability for the heat equation with singular inverse-square potentials ☆

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Abstract

We prove the null controllability of the heat equation perturbed by a singular inverse-square potential arising in quantum mechanics and combustion theory. This is done within the range of subcritical coefficients of the singular potential, provided the control acts on an annular set around the singularity. Our proof uses a splitting argument on the domain, decomposition in spherical harmonics, new Carleman inequalities and refined Hardy inequalities.

Keywords: Parabolic equation; Singular inverse-square potential; Null controllability; Carleman estimates; Observability; Hardy inequalities

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Contents

1. Introduction

Let $N \geqslant 3$ be given and consider $\Omega \subset \mathbb{R}^N$ a bounded open set such that $0 \in \Omega$ and whose boundary Γ is of class $\mathcal{C}^2$.

We analyze the controllability properties of linear heat equations with singular potentials. More precisely, we focus on the so-called *inverse-square* potential arising, for example, in the context of combustion theory [3,5,13,18] and quantum mechanics [1,12,31].

Indeed, those inverse-square potentials appear in some linearized combustion models. Consider, for instance, the semilinear elliptic equation

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega, \qquad u|_\Gamma = 0. \tag{1.1}$$

The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is assumed to be a continuous, positive, increasing and convex function with $f(0) > 0$ and $f(s)/s \to \infty$ as $s \to +\infty$. Equations like (1.1) appear in a number of applications in combustion theory, like the description of a ball of isothermal gas in gravitational equilibrium proposed by Kelvin, see [9]. Existence, uniqueness, blow-up, asymptotic behavior or stability for (1.1) or for its non-stationary version have been actively studied [5,6,13,18,19,24, 28,30]. Typical examples are $f(u)=e^u$ and $f(u)=(1+u)^p$ for some $p \geqslant 1$. In both cases (see [5, pp. 456 and 460]), there exist explicit weak solutions $u_\sharp$ (associated to some values of the parameter $\lambda_\sharp$) such that the linearized operator is of the form

$$L_\sharp = -\Delta - \lambda_\sharp f'(u_\sharp) = -\Delta - \frac{\mu}{|X|^2},$$

for some explicit constant μ.

Throughout this paper, we denote by $X=(X_1,\dots,X_N)$ the space variable in $\mathbb{R}^N$ and we keep the notation $x \in \mathbb{R}$ to represent the 1-d space variable. Moreover we use the following notation for the Euclidean norm: $|X| = (X_1^2 + \cdots + X_N^2)^{1/2}$.

Inverse-square potentials also arise in the context of quantum mechanics. For example, in [12], this type of model, involving a linear plus inversely linear electric field, is derived to analyze the confinement of neutral fermions, leading to an effective quadratic plus inversely quadratic potential in a Sturm–Liouville problem. See also [31, p. 157] for some other examples in quantum mechanics.

In this paper, we study the controllability properties of the following parabolic problem associated to this elliptic operator:

$$\begin{cases} u_t - \Delta u - \dfrac{\mu}{|X|^2} u = h\chi_\omega, & (t,X) \in (0,T)\times\Omega, \\ u(t,X) = 0, & (t,X) \in (0,T)\times\Gamma, \\ u(0,X) = u_0(X), & X \in \Omega, \end{cases} \tag{1.2}$$

with $u_0 \in L^2(\Omega)$. Here, $h \in L^2((0,T)\times\Omega)$ is the control and χ_ω stands for the characteristic set of the subdomain ω of Ω, which localizes the action of the control. The solution u of (1.2) is the state of the system.

We are concerned with the property of null controllability, i.e. whether, for all $u_0 \in L^2(\Omega)$, there exists $h \in L^2((0,T)\times\Omega)$ such that the solution u of (1.2) satisfies

$$u(T,X) \equiv 0 \quad \text{for a.e. } X \in \Omega. \tag{1.3}$$

It is well known that singular potentials of the form $V(X) = \mu/|X|^2$ generate interesting phenomena. Baras and Goldstein [1,2] discovered that existence and non-existence of positive solutions is crucially determined by the value of the parameter μ. In particular, it was proved that, for non-negative L^2 initial data and right-hand side terms, (1.2) has a unique global weak (positive) solution if $\mu \leqslant \mu^\star(N)$ whereas it has no solution, even locally in time, when $\mu > \mu^\star(N)$. Here and in what follows, $\mu^\star(N)$ stands for the critical constant

$$\mu^\star(N) := \frac{(N-2)^2}{4},$$

in the Hardy inequality guaranteeing that, for every $z \in H_0^1(\Omega)$, we have $z/|X| \in L^2(\Omega)$ and (see [22,29])

$$\forall z \in H_0^1(\Omega), \quad \mu^\star(N) \int_\Omega \frac{z^2}{|X|^2}\, dX \leqslant \int_\Omega |\nabla z|^2\, dX. \tag{1.4}$$

The work [1] generated a lot of activity on this topic and various questions have been investigated as, for example: general positive singular potentials, equations with variable coefficients, the asymptotic behavior of the solutions, semilinear equations, etc. See, for example, [7,20,21,33] and the references therein.

The value of the best constant $\mu^\star(N)$ in the Hardy inequality (1.4) plays, systematically, a crucial role when answering all these problems. In particular, (1.4) implies that, under the condition $\mu \leqslant \mu^\star(N)$, the operator $-\Delta - \mu|X|^2 I$ is nonnegative and the energy of the solutions of (1.2) decreases with time (when $h \equiv 0$). Indeed, taking $h \equiv 0$ and multiplying Eq. (1.2) by u, one obtains

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^2(t,X)\, dX &= 2\int_\Omega \left[-\left|\nabla u(t,X)\right|^2 + \mu\frac{u^2(t,X)}{|X|^2}\right] dX \\ &\leqslant 2\int_\Omega \left[-\left|\nabla u(t,X)\right|^2 + \mu^\star(N)\frac{u^2(t,X)}{|X|^2}\right] dX \leqslant 0.\end{aligned}$$

More recently, in [33], the authors complemented the results in [1] on the well-posedness of (1.2) removing the sign restriction on solutions and giving a complete description of the functional framework that we recall briefly. When $\mu < \mu^\star(N)$, $-\Delta - \mu|X|^{-2}I$ generates a coercive quadratic form in $H_0^1(\Omega)$ and this allows showing the well-posedness in the classical variational setting of the linear heat equation with smooth coefficients. However, when $\mu = \mu^\star(N)$, the space $H_0^1(\Omega)$ has to be slightly enlarged due to the logarithmic singularities of solutions at $X = 0$. (See Theorem 2.1 in Section 2.1.) Finally, when $\mu > \mu^\star(N)$, the problem is ill-posed as shown in [1].

In the present paper, we are interested in the controllability properties of such equations. In view of the results in [1,33], one may expect the null-controllability property of (1.2) to hold when $\mu \leqslant \mu^\star(N)$. Obviously, we are interested in the case where the control subdomain ω does not contain the singularity of the potential located at $X = 0$. Otherwise one can use the control (in a slightly larger class) to annihilate the effect of the singularity and to show that, whatever μ is, system (1.2) is null-controllable.

In this article, we give a complete answer in the particular case in which ω contains an annulus with center on the singularity.

We follow a, by now, well established strategy that consists in reducing the null-controllability problem to another, equivalent one, for the adjoint system in which the goal is to prove that a local measurement on the solution on ω during the time interval $0 < t < T$ provides global information everywhere in Ω. This kind of inequality is usually derived by global Carleman inequalities as developed in [17]. But the method does not apply directly in the present case because of the singularity of the potential. Indeed, standard Carleman inequalities ensure null controllability for a potential $V = V(X)$ in $L^p(\Omega)$ with $p > 2N/3$, see [23]. But this condition is not satisfied here. Therefore we adopt some of the tools developed for the analysis of the well-posedness of the initial boundary value problem and, in particular, those in [33].

The analysis in [33], based on performing a decomposition in spherical harmonics around the singularity shows that the most singular component of solutions is the radial one. The same occurs when establishing the observability inequalities. Using the fact that the observation region contains an annulus, the problem of observability can be reduced to considering a one parameter family of such problems in 1-d, the most singular one being that corresponding to the radial component.

One of the key ingredients of this article is a careful analysis of the Carleman inequalities for those 1-d problems with singular potentials, which is closely related to that in [8] and [26] on heat equations with coefficients degenerating on isolated points.

By a suitable choice of the weight in the Carleman inequality, we are able to show that the observability inequality holds if $\mu \leqslant \mu^\star(N)$. It is interesting to note that, although there is a subtle change from the subcritical ($\mu < \mu^\star(N)$) to the critical case ($\mu = \mu^\star(N)$) in what concerns well-posedness, this is not the case at the level of observability because of the strong dissipativity of the system.

As mentioned above, we treat the particular case in which ω contains an annulus with center on the singularity. In fact, as pointed out by Le Rousseau [25], the same techniques apply, in slightly more general cases in which the domain to the exterior of ω contains such an annulus, see Section 6.5. But our arguments, based on decomposition in spherical harmonics do not work for general subdomains $\omega \subset \Omega \setminus \{0\}$.

However, our approach combining spherical harmonics decomposition and 1-d Carleman estimates also yields N-d weighted Carleman estimates, see Section 6.4. Recently, these estimates have been extended by Ervedoza [14] to the case of an arbitrary nonempty open subset ω of $\Omega \setminus \{0\}$. Hence, when $\mu \leqslant \mu^\star(N)$, null controllability also holds in this more general geometric setting.

On the other hand, when $\mu > \mu^\star(N)$, the situation is not completely clear. In general, the initial-boundary value problem is not well-posed: for $u_0 \geqslant 0$ and $h \geqslant 0$, Baras and Goldstein [1] proved that there is complete and instantaneous blow-up.

However, assuming that $\mu > \mu^\star(N)$ and that Ω is a ball, the analysis developed in [33] shows that problem (1.2) is still well-posed for a subspace H_μ (defined later in Section 5) of initial conditions that oscillate sufficiently fast on the unit sphere. On the other hand, the arguments we develop here allow getting the observability of sufficiently high frequency components on the spherical harmonics decomposition. This guarantees the null controllability of (1.2) within the class of initial conditions belonging to H_μ.

But, in the supercritical case $\mu > \mu^\star(N)$, the answer is not complete. For general initial conditions and with controls h of indefinite sign, the question of whether the solution may exist and be controllable or still blows up instantaneously whatever h is, constitutes an interesting open problem.

The rest of the paper is organized as follows. In Section 2, we present our results. Next, Section 3 is devoted to the proof of the null controllability result in the subcritical and critical cases. The 1-d Carleman inequalities on which this proof is based are derived in Section 4. The supercritical case is addressed in Section 5. Finally, Section 6 is devoted to some further comments and open questions.

2. Main results

2.1. Formulation of the controllability problem

We recall that the dimension $N \in \mathbb{N}$ is such that $N \geqslant 3$. We fix an arbitrary $T > 0$ and assume that $\Omega \subset \mathbb{R}^N$ is a bounded open set such that $0 \in \Omega$ and whose boundary Γ is of class $\mathcal{C}^2$. We also use the notation $Q_T := (0, T) \times \Omega$.

Then we choose a control subdomain ω containing an annular set around the singularity, i.e. such that

$$\omega' := \{X \in \mathbb{R}^N \mid r_1 < |X| < r_2\} \subset \omega, \tag{2.1}$$

for some constants r_1, r_2 such that $0 \leqslant r_1 < r_2$, see Fig. 2.1. To fix ideas, without loss of generality, we also assume in the following that $r_2 < 1$.

Let us recall that, under the condition $\mu \leqslant \mu^\star(N)$, system (1.2) is well-posed. For the sake of simplicity, we recall the following result in [33] which makes more complete earlier results in [1].

Theorem 2.1.

(i) *Assume $\mu < \mu^\star(N)$. Then, for any $u_0 \in L^2(\Omega)$ and $h \in L^2(Q_T)$, there exists a unique weak solution of* (1.2) *such that*

$$u \in \mathcal{C}^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \qquad u_t \in L^2(0, T; H^{-1}(\Omega)).$$

(ii) *Assume $\mu = \mu^\star(N)$ and define H as being the Hilbert space obtained as the completion of $H_0^1(\Omega)$ with respect to the norm*

$$\|u\|_H = \left(\int_\Omega \left(|\nabla u|^2 - \mu^\star(N)\frac{u^2}{|X|^2}\right) dX\right)^{1/2}.$$

Then, for any $u_0 \in L^2(\Omega)$ and $h \in L^2(Q_T)$, there exists a unique weak solution of (1.2) *such that*

$$u \in \mathcal{C}^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H), \qquad u_t \in L^2(0, T; H').$$

Remark 2.1. When $\mu > \mu^\star(N)$, it was shown in [1] that problem (2.2) is not well-posed. However, by [33], there exists some subspace H_μ of sufficiently oscillating initial conditions for which the problem is well-posed. For technical reasons, the precise definition of H_μ is given later in Section 5 in which the supercritical case is addressed.

2.2. Statement of the main results

Our first main result guarantees the null controllability of system (1.2) under the condition $\mu \leqslant \mu^\star(N)$, and a partial result of null controllability for supercritical values of μ.

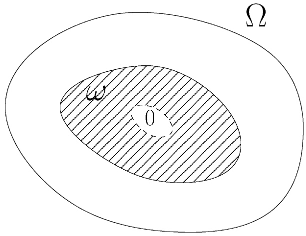

Fig. 2.1.

Theorem 2.2 *(Controllability). Assume the control subset ω fulfills the geometric condition* (2.1).

(i) *Assume $\mu \leqslant \mu^\star(N)$. Then, for all $u_0 \in L^2(\Omega)$, there exists $h \in L^2(Q_T)$ such that the solution of* (1.2) *satisfies $u(T,X) \equiv 0$ for a.e. $X \in \Omega$.*
(ii) *Assume $\mu > \mu^\star(N)$ and Ω is a ball. Then, system* (1.2) *is controllable within the class of oscillating initial data belonging to H_μ: for all $u_0 \in H_\mu$, there exists $h \in L^2(Q_T)$ such that the solution of* (1.2) *satisfies $u(T,X) \equiv 0$ for a.e. $X \in \Omega$.*

As it is classical in controllability problems and explained in the previous section, the controllability result given in point (i) of Theorem 2.2 is equivalent to an observability inequality for the adjoint system:

$$\begin{cases} v_t + \Delta v + \dfrac{\mu}{|X|^2} v = 0, & (t,X) \in Q_T, \\ v(t,X) = 0, & (t,X) \in (0,T) \times \Gamma, \\ v(T,X) = v_T(X), & X \in \Omega, \end{cases} \tag{2.2}$$

where v_T is given in $L^2(\Omega)$. More precisely, the statement of point (i) of Theorem 2.2 is equivalent to the following one:

Theorem 2.3 *(Observability). Assume* (2.1) *and $\mu \leqslant \mu^\star(N)$. Then there exists some positive constant $C_\mu = C(\mu, T, \omega) > 0$ such that, for all $v_T \in L^2(\Omega)$, the solution of* (2.2) *satisfies*

$$\int_\Omega v(0,X)^2\,dX \leqslant C_\mu \int_0^T \int_\omega v(t,X)^2\,dX\,dt. \tag{2.3}$$

Remark 2.2. As indicated in the previous section, because of the very strong dissipativity of (2.2) in the reverse sense of time, inequality (2.3) does not reflect the subtle change on the functional setting of the problem recalled in Theorem 2.2 from the case $\mu < \mu^\star(N)$ to $\mu = \mu^\star(N)$.

In the sequel we will focus on the proof of Theorem 2.3. But before doing that, we briefly recall how the controllability result in point (i) of Theorem 2.2 can be obtained from the observability inequality (2.3).

In fact, once (2.3) is known to hold, the control h whose existence is claimed in Theorem 2.2 can be taken such that $h = v^\star$ in $(0,T)\times\omega$ where $v^\star$ is the solution of (2.2) with the initial data $v_T^\star$ minimizing the following functional

$$J(v_T) = \frac{1}{2}\int_0^T\int_\omega v^2\,dX\,dt + \int_\Omega v(0,X)u_0(X)\,dX \tag{2.4}$$

in the Hilbert space $\mathcal{H}$ constituted by the initial data v_T such that the corresponding solution of (2.2) is such that

$$\int_0^T\int_\omega v^2\,dX\,dt < +\infty$$

endowed with the canonical norm

$$\|v_T\|_{\mathcal{H}} = \left[\int_0^T\int_\omega v^2\,dX\,dt\right]^{1/2}.$$

The observability inequality (2.3) guarantees that the functional $J:\mathcal{H}\to\mathbb{R}$, in addition to being continuous and convex, is coercive. This guarantees the existence of the minimizer $v_T^\star$, which is in fact unique by strict convexity. Finally it is easy to see that the fact that the differential of J at $v_T^\star$ vanishes is equivalent to the null controllability condition (1.3). We refer to [34] for more details and other applications of these arguments.

In the supercritical case, a similar argument will yield point (ii) of Theorem 2.2. This topic will be addressed in Section 5.

In the following Section 3, we prove Theorem 2.3 which ends the proof of point (i) of 2.2. The proof of point (ii) of 2.2, together with a precise definition of H_μ, is given later in Section 5.

3. Null controllability in the subcritical and critical cases

3.1. Strategy of proof of the observability inequality

In this subsection, we briefly describe the main steps of the proof of Theorem 2.3. Some of the most technical proofs will be developed in the rest of this section and Section 4.

Remark 3.1. Let us begin by a preliminary remark concerning the justification of the computations in the following proofs.

As it is classical, it is sufficient to prove (2.3) for the strong solutions v of (2.2). Then by standard density arguments, (2.3) also holds for the weak solutions v of (2.2). But, in the present situation, even the strong solutions of (2.2) do not have enough regularity to justify the computations. Indeed, for example in the case $\mu < \mu^\star(N)$, the domain of the operator is

$$D\big(-\Delta - \mu|X|^{-2}\big) = \left\{ z \in H_0^1(\Omega) \;\middle|\; -\Delta z - \frac{\mu}{|X|^2}z \in L^2(\Omega)\right\}.$$

Hence the H^2-regularity in the space variable X that is required to justify standard integrations by parts is not guaranteed. Therefore, we need to add some regularization argument to the standard procedure.

In this case this may be done by truncating or regularizing the potential. We take

$$V_n(X) = \frac{\mu}{(|X| + 1/n)^2}$$

instead of $V(X) = \mu/|X|^2$ in Eq. (2.2). The corresponding domain is $D(-\Delta - V_n) = D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ and the solutions v^n of (2.2) (with $V^n(X)$ instead of $V(X) = \mu/|X|^2$) possess all the regularity required to justify the computations. Passing to the limit as $n \to +\infty$, we recover (2.3) for the weak solutions v of (2.2).

To simplify the presentation, we directly write the computations formally for the solutions v of (2.2). They may be justified by the regularization procedure described above.

Step 1. The first step is reducing the problem to the obtention of the inequality

$$\int_{T/4}^{3T/4} \int_{\Omega} v(t, X)^2 \, dX \, dt \leqslant C_\mu \int_0^T \int_{\omega} v(t, X)^2 \, dX \, dt. \tag{3.1}$$

Indeed, according to the following lemma, (3.1) implies (2.3):

Lemma 3.1. *Assume that* (2.1) *holds and that* $\mu \leqslant \mu^\star(N)$. *If there exists some positive constant* $C_\mu = C(\mu, T, \omega) > 0$ *such that* (3.1) *holds for all* $v_T \in L^2(\Omega)$, *then there exists some other positive constant* $C_\mu = C(\mu, T, \omega) > 0$ *such that* (2.3) *also holds for all* $v_T \in L^2(\Omega)$.

The proof of this lemma is an easy consequence of the fact that, under the condition $\mu \leqslant \mu^\star(N)$, the energy of solutions of (2.2) increases with time.

Proof of Lemma 3.1. Multiplying Eq. (2.2) by v, one obtains

$$\begin{aligned}\frac{d}{dt} \int_{\Omega} v(t, X)^2 \, dX &= 2 \int_{\Omega} \left[\left|\nabla v(t, X)\right|^2 - \mu \frac{v(t, X)^2}{|X|^2} \right] dX \\ &\geqslant 2 \int_{\Omega} \left[\left|\nabla v(t, X)\right|^2 - \mu^\star(N) \frac{v(t, X)^2}{|X|^2} \right] dX \geqslant 0,\end{aligned}$$

using the fact that $\mu \leqslant \mu^\star(N)$ and Hardy inequality (1.4). Therefore the energy $\|v(t)\|^2_{L^2(\Omega)}$ of v is a non-decreasing function of t and it follows that

$$\int_{\Omega} v(0, X)^2 \, dX \leqslant \frac{2}{T} \int_{T/4}^{3T/4} \int_{\Omega} v(t, X)^2 \, dX \, dt,$$

which directly implies Lemma 3.1. □

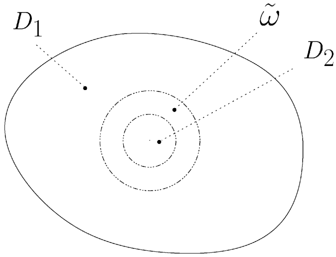

Fig. 3.1.

Step 2. Splitting of the domain. Next, to prove (3.1), we split the domain Ω on two subdomains: one containing the singularity of the potential and the other one in which the potential is bounded and smooth. In the region where the potential is bounded (i.e. near the boundary Γ), the problem may be studied by standard arguments (using classical Carleman estimates). Hence, our main task will be to treat the problem on the region where the potential is singular (i.e. near the point $X = 0$). This will be the major difficulty in our study.

For this purpose, we introduce $\tilde{r}_1, \tilde{r}_2$ such that

$$0 \leqslant r_1 < \tilde{r}_1 < \tilde{r}_2 < r_2 < 1,$$

and we set

$$\tilde{\omega} := \left\{ X \in \mathbb{R}^N \;\middle|\; \tilde{r}_1 < |X| < \tilde{r}_2 \right\} \Subset \omega' \subset \omega.$$

We also denote (see Fig. 3.1)

$$D_1 := \left\{ X \in \Omega \;\middle|\; \tilde{r}_2 < |X| \right\} \quad \text{and} \quad D_2 := \left\{ X \in \mathbb{R}^N \;\middle|\; |X| < \tilde{r}_1 \right\}.$$

Observe that

$$\Omega = \overline{D_1} \cup \tilde{\omega} \cup \overline{D_2}.$$

Obviously, inequality (3.1) can be reduced to the following two observability inequalities below:

Lemma 3.2. *Assume that* (2.1) *holds and that* $\mu \leqslant \mu^\star(N)$.

(i) *There exists some positive constant* $C_\mu = C(\mu, T, \omega) > 0$ *such that, for all* $v_T \in L^2(\Omega)$*, the solution of* (2.2) *satisfies*:

$$\int_{T/4}^{3T/4} \int_{D_1} v(t, X)^2 \, dX \, dt \leqslant C_\mu \int_0^T \int_\omega v(t, X)^2 \, dX \, dt. \tag{3.2}$$

(ii) *There exists some positive constant $C = C(T, \omega) > 0$ such that, for all $v_T \in L^2(\Omega)$, the solution of* (2.2) *satisfies*:

$$\int_{T/4}^{3T/4} \int_{D_2} v(t,X)^2 \, dX \, dt \leqslant C \int_0^T \int_\omega v(t,X)^2 \, dX \, dt. \tag{3.3}$$

Indeed, since $\Omega = \overline{D_1} \cup \tilde{\omega} \cup \overline{D_2}$ and $\tilde{\omega} \subset \omega$, the above lemma obviously implies the needed observability inequality (3.1) and therefore it proves Theorem 2.3.

Remark 3.2. The observability inequality (3.2) away from the singularity $X = 0$ is similar to the standard one that holds for the heat equation with a bounded potential term (see [15] and [17]). Hence the constant in (3.2) not only depends on T and ω but also on $\|V\|_{L^\infty(D_1)}$ i.e. on μ. On the contrary, the constant that appears in the observability inequality (3.3) near the singularity only depends on T and ω but is independent of $\mu \leqslant \mu^\star(N)$.

To rigorously prove Lemma 3.2, we use a standard cut-off argument. We introduce two non-negative cut-off functions $\phi_1, \phi_2 : \overline{\Omega} \to \mathbb{R}$ of class $\mathcal{C}^\infty$ such that

$$\begin{cases} \forall X \in \tilde{\omega}, & 0 < \phi_1(X) < 1, \\ \forall X \in D_1, & \phi_1(X) = 1, \\ \forall X \in D_2, & \phi_1(X) = 0, \end{cases} \qquad \begin{cases} \forall X \in \tilde{\omega}, & 0 < \phi_2(X) < 1, \\ \forall X \in D_1, & \phi_2(X) = 0, \\ \forall X \in D_2, & \phi_2(X) = 1. \end{cases}$$

Next, for $i = 1, 2$, we define $v_i := \phi_i v$ and we notice that v_i satisfies

$$\begin{cases} v_{i,t} + \Delta v_i + \dfrac{\mu}{|X|^2} v_i = g_i, & (t,X) \in Q_T, \\ v_i(t,X) = 0, & (t,X) \in (0,T) \times \Gamma, \\ v_i(T,X) = v_{i,T}(X), & X \in \Omega, \end{cases}$$

where $v_{i,T} := \phi_i v_T$ and

$$g_i := \Delta\phi_i v + 2\nabla\phi_i \cdot \nabla v.$$

We now derive (3.2) and (3.3) applying Carleman inequalities to v_1 and v_2. Of course, the main difficulties arise when working on the subdomain D_2 where the singularity is located.

Step 3. The observability inequality away from the singularity. Observe that ϕ_1 has its support in $D := \tilde{\omega} \cup \overline{D_1} = \{X \in \Omega \mid |X| > \tilde{r}_1\}$ which does not contain the singularity. Hence v_1 solves

$$\begin{cases} v_{1,t} + \Delta v_1 = \Delta\phi_1 v + 2\nabla\phi_1 \cdot \nabla v - \dfrac{\mu}{|X|^2}\phi_1 v, & (t,X) \in (0,T) \times D, \\ v_1(t,X) = 0, & (t,X) \in (0,T) \times \partial D, \\ v_1(T,X) = v_{1,T}(X), & X \in D. \end{cases} \tag{3.4}$$

Note that, because of the fact that $0 \notin D$, the potential $\mu|X|^{-2}$ is bounded in D and consequently the right-hand side term in (3.4) belongs to $L^2((0,T) \times D)$. Hence we can apply standard Carleman estimates [15,17] to get (3.2). The details of the proof of (3.2) are developed in Section 3.2.

Step 4. The observability inequality near the singularity. On the other hand, ϕ_2 has its support in $\overline{D_2} \cup \tilde{\omega} = \{X \in \mathbb{R}^N \mid |X| < \tilde{r}_2\}$. In particular, since $\tilde{r}_2 < 1$, ϕ_2 is supported in $\mathbb{B}^N$ where $\mathbb{B}^N$ stands for the unit ball of $\mathbb{R}^N$. We have

$$\begin{cases} v_{2,t} + \Delta v_2 + \dfrac{\mu}{|X|^2} v_2 = g_2, & (t, X) \in (0, T) \times \mathbb{B}^N, \\ v_2(t, X) = 0, & (t, X) \in (0, T) \times \partial\mathbb{B}^N, \\ v_2(T, X) = v_{2,T}(X), & X \in \mathbb{B}^N. \end{cases} \tag{3.5}$$

On the contrary to (3.2), (3.3) is much more delicate to obtain since the potential is singular at $X = 0$. For this reason, one cannot use standard Carleman inequalities anymore. The proof of (3.3) requires some new arguments such as the decomposition on spherical harmonics (used in [33] to analyze well-posedness) and *new Carleman estimates adapted to the singularity of the potential*. The details of the proof of (3.3) are developed in Section 3.3.

3.2. Observability estimate away from the singularity

3.2.1. Proof of (3.2)

As mentioned in Section 3.1, we are in the classical frame in which standard Carleman estimates can be applied to v_1.

Let us recall the standard Carleman estimate for the heat operator in a domain D with measurements in a subdomain ω. Following [17] (see also [15]), we introduce a function $\eta^0 : \overline{D} \to \mathbb{R}$ of class $\mathcal{C}^2$ such that

$$\eta^0 > 0 \quad \text{in } D, \qquad \eta^0 = 0 \quad \text{on } \partial D, \quad \text{and} \quad \nabla\eta^0 \neq 0 \quad \text{in } \overline{D \setminus \omega}.$$

Next we consider $K_0 > 0$ such that $K_0 \geqslant 5 \max_{\overline{D}} \eta^0$ and we set

$$\beta^0 := \eta^0 + K_0, \quad \bar{\beta} := \frac{5}{4} \max_{\overline{D}} \beta^0 \quad \text{and} \quad \rho^1(X) := e^{S\bar{\beta}} - e^{S\beta^0(X)} \quad \text{for all } X \in \overline{D},$$

where S is a sufficiently large positive constant (that only depends on D and ω). Notice that $\rho^1 > 0$ in $\overline{D}$. We also introduce

$$\forall (t, X) \in (0, T) \times \overline{D}, \quad \rho(t, X) := \exp\left(\frac{\rho^1(X)}{t(T - t)}\right).$$

Then the following result holds:

Theorem 3.1. *(See [17].) Let $D \subset \mathbb{R}^N$ be a bounded open set whose boundary is of class $\mathcal{C}^2$. Consider ω a nonempty open subset of D and set $T > 0$ and the weight function ρ as above. Then there exists $C_\star, s_1 > 0$ such that, for all $s \geqslant s_1$,*

$$\frac{1}{s} \int_0^T \int_D \rho^{-2s} t(T - t)\big(|q_t|^2 + |\Delta q|^2\big)\, dX\, dt$$

$$+ s \int_0^T \int_D \rho^{-2s} t^{-1}(T-t)^{-1} |\nabla q|^2 \, dX \, dt + s^3 \int_0^T \int_D \rho^{-2s} t^{-3}(T-t)^{-3} |q|^2 \, dX \, dt$$

$$\leqslant C_\star \left[\int_0^T \int_D \rho^{-2s} |q_t + \Delta q|^2 \, dX \, dt + s^3 \int_0^T \int_\omega \rho^{-2s} t^{-3}(T-t)^{-3} |q|^2 \, dX \, dt \right],$$

for all $q \in \mathcal{C}^2((0,T) \times \overline{D})$ *such that* $q = 0$ *on* $(0,T) \times \partial D$.

We now apply Theorem 3.1 with $q = v_1$ and obtain

$$s^3 \int_0^T \int_D \rho^{-2s} t^{-3}(T-t)^{-3} |\phi_1 v|^2 \leqslant C_\star \int_0^T \int_D \rho^{-2s} \left| \Delta\phi_1 v + 2\nabla\phi_1 \cdot \nabla v - \frac{\mu}{|X|^2} \phi_1 v \right|^2$$

$$+ C_\star s^3 \int_0^T \int_\omega \rho^{-2s} t^{-3}(T-t)^{-3} |\phi_1 v|^2,$$

for all $s \geqslant s_1$. Using the fact that $\phi_1 \equiv 1$ in D_1 and that $0 \leqslant \phi_1 \leqslant 1$ everywhere, together with the fact that $\nabla\phi_1$ and $\Delta\phi_1$ are bounded and with support in $\tilde{\omega}$, we deduce that

$$s^3 \int_0^T \int_{D_1} \rho^{-2s} t^{-3}(T-t)^{-3} v^2 \leqslant C \int_0^T \int_{\tilde{\omega}} \rho^{-2s} \left(v^2 + |\nabla v|^2\right) + C \int_0^T \int_D \rho^{-2s} \frac{\mu^2}{|X|^4} v^2$$

$$+ C_\star s^3 \int_0^T \int_\omega \rho^{-2s} t^{-3}(T-t)^{-3} v^2$$

$$\leqslant C_s \int_0^T \int_\omega v^2 + C \int_0^T \int_{\tilde{\omega}} \rho^{-2s} |\nabla v|^2 + C \int_0^T \int_D \rho^{-2s} \frac{\mu^2}{|X|^4} v^2.$$

On the other hand, since $|X| > \tilde{r}_1$ on D, we can write

$$\int_0^T \int_D \rho^{-2s} \frac{\mu^2}{|X|^4} v^2 \leqslant \frac{\mu^2}{\tilde{r}_1^4} \int_0^T \int_D \rho^{-2s} v^2$$

$$\leqslant C\mu^2 \int_0^T \int_{\tilde{\omega}} \rho^{-2s} v^2 + C\mu^2 \int_0^T \int_{D_1} \rho^{-2s} v^2$$

$$\leqslant C\mu^2 \int_0^T \int_\omega v^2 + C \frac{\mu^2}{s^3} s^3 \int_0^T \int_{D_1} \rho^{-2s} t^{-3}(T-t)^{-3} v^2.$$

Hence

$$s^3\left(1 - C\frac{\mu^2}{s^3}\right)\int_0^T\int_{D_1} \rho^{-2s}t^{-3}(T-t)^{-3}v^2 \leqslant \left(C_s + C\mu^2\right)\int_0^T\int_{\omega} v^2 + C\int_0^T\int_{\tilde{\omega}} \rho^{-2s}|\nabla v|^2.$$

At this stage, it remains to use the following Caccioppoli's inequality to estimate the last quantity of the right-hand side of the above inequality in terms of the first one:

Lemma 3.3 *(Caccioppoli's inequalities). Assume that* (2.1) *and* $\mu \leqslant \mu^\star(N)$ *hold. Let* $\tilde{\sigma} : (0,T) \times \overline{\Omega} \to \mathbb{R}_+^*$ *be a function of the form*

$$\tilde{\sigma}(t,X) = p(X)\theta(t)$$

where $p : \overline{\Omega} \to \mathbb{R}_+^*$ *is a smooth nonnegative function and where* $\theta : (0,T) \to \mathbb{R}_+^*$ *is defined by*

$$\theta(t) = \left(\frac{1}{t(T-t)}\right)^k$$

for some $k \geqslant 1$. *Then there exists some* $C > 0$ (*independent of* μ) *such that, for all* $v_T \in L^2(\Omega)$, *the solution of* (2.2) *satisfies*

$$\int_0^T\int_{\tilde{\omega}} \left|\nabla v(t,X)\right|^2 e^{-\tilde{\sigma}(t,X)}\, dX\, dt \leqslant C\int_0^T\int_{\omega} v(t,X)^2\, dX\, dt.$$

The proof of this lemma is given in Section 3.2.2 below.

Applying Lemma 3.3 with $\tilde{\sigma}(t,X) = 2s\rho^1(X)/(t(T-t))$, we obtain

$$\int_0^T\int_{\tilde{\omega}} \rho^{-2s}|\nabla v|^2 = \int_0^T\int_{\tilde{\omega}} |\nabla v|^2 e^{-\tilde{\sigma}(t,X)} \leqslant C\int_0^T\int_{\omega} v^2.$$

Hence, fixing $s = s_\mu$ large enough (in a way that depends on μ), we get

$$\int_0^T\int_{D_1} \rho^{-2s_\mu}t^{-3}(T-t)^{-3}v^2 \leqslant C_\mu\int_0^T\int_{\omega} v^2,$$

for some constant $C_\mu > 0$. Since $\rho^{-2s_\mu}t^{-3}(T-t)^{-3} \geqslant c_\mu > 0$ on $(T/4, 3T/4)\times\Omega$, we conclude that

$$\int_{T/4}^{3T/4}\int_{D_1} v^2 \leqslant C_\mu\int_0^T\int_{\omega} v^2,$$

for some other constant $C_\mu > 0$, as we wanted to prove.

3.2.2. Proof of Lemma 3.3

Let us recall that ω and $\tilde{\omega}$ satisfy $\tilde{\omega} \Subset \omega$ and let us consider a smooth function $\xi : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} 0 \leqslant \xi(X) \leqslant 1, & \text{for all } X \in \overline{\Omega}, \\ \xi(X) = 1, & \text{for } X \in \tilde{\omega}, \\ \xi(X) = 0, & \text{for } X \notin \omega. \end{cases}$$

By assumption on $\tilde{\sigma}$, we have $e^{-\tilde{\sigma}(t,\cdot)} \equiv 0$ for $t = 0$ and $t = T$. Hence we get

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt} \int_\Omega \xi^2 e^{-\tilde{\sigma}} v^2 = \iint_{Q_T} -\xi^2 \tilde{\sigma}_t e^{-\tilde{\sigma}} v^2 + 2\xi^2 e^{-\tilde{\sigma}} v v_t \\ &= -\iint_{Q_T} \xi^2 \tilde{\sigma}_t e^{-\tilde{\sigma}} v^2 - 2 \iint_{Q_T} \xi^2 e^{-\tilde{\sigma}} v \Delta v - 2\mu \iint_{Q_T} \xi^2 e^{-\tilde{\sigma}} \frac{v^2}{|X|^2}, \end{aligned}$$

where we used Eq. (2.2) satisfied by v. It follows that

$$\begin{aligned} 0 &= -\iint_{Q_T} \xi^2 \tilde{\sigma}_t e^{-\tilde{\sigma}} v^2 + 2 \iint_{Q_T} \nabla\big(\xi^2 e^{-\tilde{\sigma}} v\big) \cdot \nabla v - 2\mu \iint_{Q_T} \xi^2 e^{-\tilde{\sigma}} \frac{v^2}{|X|^2} \\ &= -\iint_{Q_T} \xi^2 \tilde{\sigma}_t e^{-\tilde{\sigma}} v^2 + 2 \iint_{Q_T} v \nabla\big(\xi^2 e^{-\tilde{\sigma}}\big) \cdot \nabla v + 2 \iint_{Q_T} \xi^2 e^{-\tilde{\sigma}} |\nabla v|^2 - 2\mu \iint_{Q_T} \xi^2 e^{-\tilde{\sigma}} \frac{v^2}{|X|^2}. \end{aligned}$$

Hence,

$$\begin{aligned} 2 \iint_{Q_T} \xi^2 e^{-\tilde{\sigma}} |\nabla v|^2 &= \iint_{Q_T} \xi^2 \tilde{\sigma}_t e^{-\tilde{\sigma}} v^2 - 2 \iint_{Q_T} v \nabla\big(\xi^2 e^{-\tilde{\sigma}}\big) \cdot \nabla v + 2\mu \iint_{Q_T} \xi^2 e^{-\tilde{\sigma}} \frac{v^2}{|X^2|} \\ &= \iint_{Q_T} \xi^2 \tilde{\sigma}_t e^{-\tilde{\sigma}} v^2 - 2 \iint_{Q_T} \big(\xi e^{-\tilde{\sigma}/2} \nabla v\big) \cdot \left(\frac{\nabla(\xi^2 e^{-\tilde{\sigma}})}{\xi e^{-\tilde{\sigma}/2}} v \right) \\ &\quad + 2\mu \iint_{Q_T} \xi^2 e^{-\tilde{\sigma}} \frac{v^2}{|X|^2}. \end{aligned}$$

Using also the assumption $\mu \leqslant \mu^\star(N)$, we deduce

$$\begin{aligned} 2 \iint_{Q_T} \xi^2 e^{-\tilde{\sigma}} |\nabla v|^2 \leqslant{} & \iint_{Q_T} \xi^2 \tilde{\sigma}_t e^{-\tilde{\sigma}} v^2 + \iint_{Q_T} \xi^2 e^{-\tilde{\sigma}} |\nabla v|^2 \\ & + \iint_{Q_T} \left| \frac{\nabla(\xi^2 e^{-\tilde{\sigma}})}{\xi e^{-\tilde{\sigma}/2}} \right|^2 v^2 + \frac{(N-2)^2}{2} \iint_{Q_T} \xi^2 e^{-\tilde{\sigma}} \frac{v^2}{|X|^2}. \end{aligned}$$

It follows that

$$\iint_{Q_T} \xi^2 e^{-\tilde\sigma} |\nabla v|^2 \leqslant \iint_{Q_T} \xi^2 \tilde\sigma_t e^{-\tilde\sigma} v^2 + \iint_{Q_T} \left| \frac{\nabla(\xi^2 e^{-\tilde\sigma})}{\xi e^{-\tilde\sigma/2}} \right|^2 v^2 + \frac{(N-2)^2}{2} \iint_{Q_T} \xi^2 e^{-\tilde\sigma} \frac{v^2}{|X|^2}.$$

Using the assumptions on $\tilde\sigma$, p and θ, one can see that the functions

$$(t,X) \mapsto \left| \frac{\nabla(\xi^2 e^{-\tilde\sigma})}{\xi e^{-\tilde\sigma/2}} \right| = 2\nabla\xi e^{-p\theta/2} - \xi \nabla p \theta e^{-p\theta/2}$$

and

$$(t,X) \mapsto \xi^2 e^{-\tilde\sigma} \frac{1}{|X|^2}$$

are bounded on $Q_T = (0,T) \times \Omega$ with support in $(0,T) \times \omega$. In the same spirit, we use the fact $|\theta_t(t)| \leqslant C\theta(t)^{1+1/k}$ to say that

$$(t,X) \mapsto \xi^2 \tilde\sigma_t e^{-\tilde\sigma} = \xi^2 p\theta_t e^{-p\theta}$$

is also bounded on $Q_T = (0,T) \times \Omega$ with support in $(0,T) \times \omega$. Therefore, there exists some constant $C > 0$ such that

$$\iint_{Q_T} \xi^2 e^{-\tilde\sigma} |\nabla v|^2 \leqslant C \int_0^T \int_\omega v^2.$$

Notice that this constant $C > 0$ depends on $\tilde\sigma$ but is independent of μ (provided that μ satisfies the condition $\mu \leqslant \mu^\star(N)$). Finally, since $\xi \equiv 1$ in $\tilde\omega$, we get

$$\int_0^T \int_{\tilde\omega} |\nabla v|^2 e^{-\tilde\sigma} \leqslant \int_0^T \int_\Omega \xi^2 e^{-\tilde\sigma} |\nabla v|^2 \leqslant C \int_0^T \int_\omega v^2.$$

3.3. Observability estimate near the singularity

We proceed as follows. First, in Section 3.3.1, we decompose the N-d problem (2.2) on spherical harmonics. Then the proof of (3.3) is reduced to proving some *uniform* Carleman estimates for a infinite family of 1-d singular parabolic equations. Section 3.3.2 is devoted to the statement of this result which consists precisely on guaranteeing that the needed Carleman inequalities hold uniformly on the value of the singular coefficient. Finally, in Section 3.3.3, we apply them to the infinite family of 1-d singular problems to conclude the proof of (3.3).

3.3.1. Decomposition on spherical harmonics

Now it remains to prove (3.3). As mentioned in Section 3.1, v_2 solves (3.5) in $\mathbb{B}^N$ and therefore we can apply a decomposition on spherical harmonics.

We consider the diffeomorphism (see [10, Chapter 2, Section 1.4])

$$\begin{cases} \mathbb{R}^N \setminus \{0\} \to (0, +\infty) \times \mathbb{S}^{N-1}, \\ X \mapsto (r, \sigma) := \left(|X|, \dfrac{X}{|X|}\right) \end{cases}$$

where $\mathbb{S}^{N-1}$ is the unit sphere in $\mathbb{R}^N$. Let us recall that for any $f \in L^1(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} f(X)\, dX = \int_0^{+\infty} r^{N-1}\, dr \int_{\mathbb{S}^{N-1}} f(r\sigma)\, d\sigma$$

where $d\sigma$ denotes the surface measure on $\mathbb{S}^{N-1}$.

On the other hand, we introduce the Laplace–Beltrami operator Δ_σ defined by (see [10, Chapter 2, Section 1.4]):

$$\Delta_\sigma g(\sigma) = \Delta\left(g\left(\frac{X}{|X|}\right)\right)_{|X|=1}$$

for any function g defined on $\mathbb{S}^{N-1}$. Then we can write the Laplacian in spherical coordinates (see [10, Chapter 2, Section 1.4]):

$$\Delta v = \frac{\partial^2 v}{\partial r^2} + \frac{N-1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\Delta_\sigma v.$$

Now we rewrite problem (3.5) in spherical coordinates. We denote

$$\bar{v}(t, r, \sigma) = v_2(t, r\sigma), \qquad \bar{v}_T(r, \sigma) = v_{2,T}(r\sigma) \quad \text{and} \quad \bar{g}(t, r, \sigma) = g_2(t, r\sigma).$$

Since v_2 is supported in $(0, T) \times (\overline{D_2} \cup \tilde{\omega})$, we have $\bar{v}(t, r, \sigma) = 0$ for all $r \in (\tilde{r}_2, 1)$. Then, using (3.5), $\bar{v}$ satisfies

$$\begin{cases} \bar{v}_t + \bar{v}_{rr} + \dfrac{N-1}{r}\bar{v}_r + \dfrac{1}{r^2}\Delta_\sigma \bar{v} + \dfrac{\mu}{r^2}\bar{v} = \bar{g}, & (t, r, \sigma) \in (0, T) \times (0, 1) \times \mathbb{S}^{N-1}, \\ \bar{v}(t, r, \sigma) = 0, & (t, r, \sigma) \in (0, T) \times (\tilde{r}_2, 1) \times \mathbb{S}^{N-1}, \\ \bar{v}(T, r, \sigma) = \bar{v}_T(r, \sigma), & (r, \sigma) \in (0, 1) \times \mathbb{S}^{N-1}. \end{cases} \tag{3.6}$$

Since $v_{2,T} \in L^2(\mathbb{B}^N)$, $\bar{v}_T$ satisfies

$$\int_{\mathbb{S}^{N-1}} \int_0^1 \bar{v}_T(r, \sigma)^2 r^{N-1}\, dr\, d\sigma < +\infty. \tag{3.7}$$

Now we decompose (3.6) into spherical harmonics. Let us recall that the eigenvalues of the Laplace–Beltrami operator $-\Delta_\sigma$ are given by (see [10, Chapter 8, Section 8.1.4] in the case $N=3$ or [4,32] for the general case):

$$\forall k \geqslant 0, \quad d_k = k(N+k-2).$$

Moreover $L^2(\mathbb{S}^{N-1}) = \bigoplus_{k\geqslant 0} V_k$ where V_k is the eigenspace associated to the eigenvalue d_k. For each $k \geqslant 0$, we denote by l_k the dimension of V_k, and by $(f^{k,l})_{1\leqslant l\leqslant l_k}$ an orthonormal basis of V_k. Finally, $(f^{k,l})_{k\geqslant 0,\,1\leqslant l\leqslant l_k}$ forms an orthonormal basis of $L^2(\mathbb{S}^{N-1})$ such that

$$\forall k \geqslant 0,\ 1 \leqslant l \leqslant l_k, \quad -\Delta_\sigma f^{k,l} = d_k f^{k,l}.$$

Hence we decompose $\bar{v}$, $\bar{v}_T$ and $\bar{g}$ into spherical harmonics as follows:

$$\bar{v}(t,r,\sigma) = \sum_{k,l} v^{k,l}(t,r) f^{k,l}(\sigma), \qquad \bar{v}_T(r,\sigma) = \sum_{k,l} v_T^{k,l}(r) f^{k,l}(\sigma),$$

$$\bar{g}(t,r,\sigma) = \sum_{k,l} g^{k,l}(t,r) f^{k,l}(\sigma).$$

It follows from (3.6) that, for all k,l, the function $v^{k,l}$ solves the following 1-d problem:

$$\begin{cases} v_t^{k,l} + v_{rr}^{k,l} + \dfrac{N-1}{r} v_r^{k,l} + \dfrac{\mu - d_k}{r^2} v^{k,l} = g^{k,l}, & (t,r) \in (0,T) \times (0,1), \\ v^{k,l}(t,r) = 0, & (t,r) \in (0,T) \times (\tilde{r}_2, 1), \\ v^{k,l}(T,r) = v_T^{k,l}(r), & r \in (0,1). \end{cases} \tag{3.8}$$

By (3.7), we have

$$\sum_{k,l} \int_0^1 v_T^{k,l}(r)^2 r^{N-1}\, dr < +\infty.$$

In particular $v_T^{k,l} \in L^2_{N-1}(0,1)$ for all k,l, where $L^2_{N-1}(0,1)$ stands for the weighted L^2-space:

$$L^2_{N-1}(0,1) := \left\{ z : (0,1) \to \mathbb{R} \ \text{measurable} \ \Big| \int_0^1 z(r)^2 r^{N-1}\, dr < +\infty \right\}.$$

Next we show that each problem (3.8) may be transformed into a simpler one. Indeed, let us set

$$\tilde{v}^{k,l}(t,r) := r^{(N-1)/2} v^{k,l}(t,r), \qquad \tilde{v}_T^{k,l}(r) := r^{(N-1)/2} v_T^{k,l}(r),$$
$$\tilde{g}^{k,l}(t,r) := r^{(N-1)/2} g^{k,l}(t,r). \tag{3.9}$$

Note that $\tilde{v}^{k,l}(t,0)=0$ since $N \geqslant 3$. Moreover $\tilde{v}^{k,l}(t,r)=0$ for all $r \in (\tilde{r}_2, 1)$. Hence, for all k,l, the function $\tilde{v}^{k,l}$ satisfies

$$\begin{cases} \tilde{v}^{k,l}_t + \tilde{v}^{k,l}_{rr} + \dfrac{\lambda_k}{r^2}\tilde{v}^{k,l} = \tilde{g}^{k,l}, & (t,r) \in (0,T)\times(0,1),\\ \tilde{v}^{k,l}(t,0)=0, & t \in (0,T),\\ \tilde{v}^{k,l}(t,r)=0, & (t,r)\in(0,T)\times(\tilde{r}_2,1),\\ \tilde{v}^{k,l}(T,r)=\tilde{v}^{k,l}_T(r), & r\in(0,1), \end{cases} \tag{3.10}$$

where

$$\lambda_k := \mu - \frac{(N-1)(N-3)}{4} - d_k \tag{3.11}$$

and where $\tilde{v}^{k,l}_T$ belongs to $L^2(0,1)$ (since $v^{k,l}_T \in L^2_{N-1}(0,1)$).

Let us observe that, using the fact that $\mu \leqslant \mu^\star(N)$ and $d_k \geqslant 0$, we have, for all $k \geqslant 0$:

$$\lambda_k \leqslant \mu^\star(N) - \frac{(N-1)(N-3)}{4} \leqslant \frac{(N-2)^2}{4} - \frac{(N-1)(N-3)}{4} = \frac{1}{4}.$$

Hence the values of λ_k in (3.10) correspond to subcritical or critical parameters in dimension 1 satisfying

$$\forall k \geqslant 0, \quad \lambda_k \leqslant \mu^\star(1) = \frac{1}{4}. \tag{3.12}$$

At this stage, we need to derive new Carleman estimates for 1-d singular problems like (3.10) with constants that are independent of k and l. Hence, in the following, we concentrate on the following 1-d equation

$$\begin{cases} w_t + w_{xx} + \dfrac{\lambda}{x^2}w + \dfrac{m}{x^\beta}w = f, & (t,x)\in Q_T,\\ w(t,0)=w(t,1)=0, & t\in(0,T),\\ w(T,x)=w_T(x), & x\in(0,1). \end{cases} \tag{3.13}$$

Note that, in the present context, the space variable is denoted by x, $Q_T := (0,T)\times(0,1)$, $w_T \in L^2(0,1)$ and $f \in L^2(Q_T)$.

We consider more general systems of this form since the same proofs apply to them. In the particular case where $m=0$, we recover the systems above (3.10).

3.3.2. Carleman estimates for 1-d singular problems

As mentioned previously, one of the main contributions of this paper is to derive new Carleman estimates for the singular 1-d problem (3.13). At this point it is convenient to recall that the Hardy inequality (1.4) also holds in dimension $N=1$ (see for instance [11, Chapter 5.3]):

$$\forall z \in H^1_0(0,1), \quad \int_0^1 \frac{z^2}{x^2}\,dx \leqslant 4\int_0^1 z_x^2\,dx. \tag{3.14}$$

We restrict our study to the solutions of (3.13) that vanish in a neighbourhood of $x = 1$ since this condition is automatically satisfied in the application to the proof of (3.3) by the cut-off construction. More precisely, we consider the solutions of (3.13) satisfying

$$w(t, x) = 0 \quad \text{for all } (t, x) \in (0, T) \times (1 - \eta, 1) \quad \text{for some } 0 < \eta < 1. \tag{3.15}$$

The following Carleman inequality holds for these solutions of (3.13):

Theorem 3.2 *(Singular Carleman estimates). Assume that $\lambda \leqslant 1/4$, $0 \leqslant \beta < 2$ and $m \in \mathbb{R}$. For every $\gamma < 2$, consider the function $\sigma : (0, T) \times [0, 1] \to \mathbb{R}_+$ defined by*

$$\sigma(t, x) := \theta(t)\left(1 - \frac{x^2}{2}\right) \quad \textit{where } \theta(t) := \left(\frac{1}{t(T - t)}\right)^k, \quad k := 1 + \frac{2}{\gamma}.$$

Then, there exists $R_0 > 0$ such that, for all $R \geqslant R_0$, the following inequality holds

$$R^3 \iint\limits_{Q_T} \theta^3 x^2 w^2 e^{-2R\sigma} + 2R\left(\frac{1}{4} - \lambda\right) \iint\limits_{Q_T} \theta \frac{w^2}{x^2} e^{-2R\sigma} + \frac{R}{2} \iint\limits_{Q_T} \theta \frac{w^2}{x^\gamma} e^{-2R\sigma}$$
$$\leqslant \frac{1}{2} \iint\limits_{Q_T} f^2 e^{-2R\sigma},$$

for the solutions w of (3.13) *satisfying condition* (3.15).

The proof of Theorem 3.2 is given in Section 4.

Remark 3.3. This Carleman inequality provides, in addition to the estimate of the weighted L^2-norm of w, also an estimate on w^2/x^γ. It also provides an estimate of w^2/x^2 but with a constant factor that vanishes as $\lambda \to 1/4$, and this is natural in view of the fact that it corresponds to the limit case in the Hardy inequality.

Remark 3.4. It is easy to see that Theorem 3.2 implies null controllability results for some singular 1-d problems, see Section 6. Moreover, in the case where the domain Ω is a ball, by means of the decomposition in spherical harmonics, Theorem 3.2 also provides other new Carleman estimates in N-d that are given in Section 6.

Remark 3.5. Instead of Theorem 3.2, taking into account that the singular potentials are actually smooth near $x = 1$, one could also prove Carleman estimates that hold for all solutions of (3.13) (without the condition (3.15)). But the constants appearing in those estimates would strongly depend on λ and blow up when $\lambda \to -\infty$. Hence they are not sharp enough to ensure the key property that the constant remains uniformly bounded for all $\lambda \leqslant 1/4$ as it is required for our goal.

3.3.3. Proof of (3.3)

In this last step, we apply Theorem 3.2 to obtain an uniform observability inequality for the infinite family of 1-d singular problems (3.10). Then we prove that it implies (3.3).

Let us recall that, by (3.12), $\lambda_k \leqslant 1/4$ for all $k \geqslant 0$. Moreover, by changing notations ($r \to x$) and using (3.10), we see that, for all k, l, $w = \tilde{v}^{k,l}$ solves (3.13) for $\lambda = \lambda_k$, $m = 0$ and $f = \tilde{g}^{k,l}$ and that it also satisfies (3.15). Therefore we are in the frame in which Theorem 3.2 can be applied. We do it by taking $\gamma = 0$.

For all k, l and all $R \geqslant R_0$, we obtain

$$\frac{R}{2} \iint_{Q_T} \theta \big(\tilde{v}^{k,l}\big)^2 e^{-2R\sigma} \leqslant \frac{1}{2} \iint_{Q_T} \big(\tilde{g}^{k,l}\big)^2 e^{-2R\sigma}.$$

Using the definition of σ, one can check that there exists some constant $C_R > 0$ such that

$$\theta(t) e^{-2R\sigma(t,r)} \geqslant C_R \quad \text{for all } (t,r) \in (T/4, 3T/4) \times (0,1).$$

Therefore,

$$C_R R \int_{T/4}^{3T/4} \int_0^1 \big(\tilde{v}^{k,l}\big)^2 \leqslant \iint_{Q_T} \big(\tilde{g}^{k,l}\big)^2 e^{-2R\sigma}.$$

Let us fix R such that $R \geqslant R_0$. Then

$$\int_{T/4}^{3T/4} \int_0^1 \big(\tilde{v}^{k,l}\big)^2 \, dr\, dt \leqslant C \int_0^T \int_0^1 \big(\tilde{g}^{k,l}\big)^2 e^{-2R\sigma} \, dr\, dt,$$

for some constant $C > 0$ that is independent of k and l.

By the change of variables (3.9), this becomes

$$\int_{T/4}^{3T/4} \int_0^1 \big(v^{k,l}\big)^2 r^{N-1} \, dr\, dt \leqslant C \int_0^T \int_0^1 \big(g^{k,l}\big)^2 e^{-2R\sigma(t,r)} r^{N-1} \, dr\, dt.$$

Hence

$$\int_{T/4}^{3T/4} \int_0^1 \bigg(\sum_{k,l} \big(v^{k,l}\big)^2\bigg) r^{N-1} \, dr\, dt \leqslant C \int_0^T \int_0^1 \bigg(\sum_{k,l} \big(g^{k,l}\big)^2\bigg) e^{-2R\sigma(t,r)} r^{N-1} \, dr\, dt.$$

It follows that

$$\int_{T/4}^{3T/4} \int_0^1 \int_{\mathbb{S}^{N-1}} \bar{v}(t,r,\sigma)^2 r^{N-1} \, d\sigma\, dr\, dt \leqslant C \int_0^T \int_0^1 \int_{\mathbb{S}^{N-1}} \bar{g}(t,r,\sigma)^2 e^{-2R\sigma(t,r)} r^{N-1} \, d\sigma\, dr\, dt.$$

And finally, we obtain

$$\int_{T/4}^{3T/4}\int_{\mathbb{B}^N} v_2(t,X)^2\,dX\,dt \leqslant C\int_0^T\int_{\mathbb{B}^N} g_2(t,X)^2 e^{-2R\sigma(t,|X|)}\,dX\,dt.$$

By definition of v_2 and g_2 this becomes

$$\int_{T/4}^{3T/4}\int_{\mathbb{B}^N} |\phi_2 v|^2\,dX\,dt \leqslant C\int_0^T\int_{\mathbb{B}^N} |\Delta\phi_2 v + 2\nabla\phi_2\cdot\nabla v|^2 e^{-2R\sigma(t,|X|)}\,dX\,dt.$$

Next, using the fact that $\phi_2 \equiv 1$ in D_2 and the fact that $\nabla\phi_2$ and $\Delta\phi_2$ are bounded and supported in $\tilde{\omega} \subset \omega$, we deduce

$$\begin{aligned}\int_{T/4}^{3T/4}\int_{D_2} v^2\,dX\,dt &\leqslant C\int_0^T\int_{\tilde{\omega}} \big(v^2 + |\nabla v|^2\big)e^{-2R\sigma(t,|X|)}\,dX\,dt\\ &\leqslant C\int_0^T\int_{\omega} v^2\,dX\,dt + \int_0^T\int_{\tilde{\omega}} |\nabla v|^2 e^{-2R\sigma(t,|X|)}\,dX\,dt.\end{aligned}$$

At this stage, it remains to use the Caccioppoli's inequality given by Lemma 3.3 to estimate the last quantity of the right-hand side of the above inequality in terms of the first one. Applying Lemma 3.3 with $\tilde{\sigma}(t,X) = 2R\sigma(t,|X|)$, we obtain

$$\int_{T/4}^{3T/4}\int_{D_2} v^2\,dX\,dt \leqslant C\int_0^T\int_{\omega} v^2\,dX\,dt,$$

which ends the proof of (3.3).

4. 1-d Carleman inequalities

This section is devoted to the proof of Theorem 3.2. In the first subsection, we describe the main steps of the proof whereas the next subsections contain the technical parts of the proof.

Remark 4.1. As for the proof of Theorem 2.3 (see Remark 3.1), we write here formal computations for the solutions w of (3.13). However they can be justified following the regularization procedure described in Remark 3.1 taking a potential $\lambda(x+1/n)^{-2}$ instead of λx^{-2} in (3.13).

4.1. Outline of the proof

With no loss of generality, we first assume that $\beta < \gamma < 2$. Indeed it is sufficient to prove the result for all γ such that $\beta < \gamma < 2$ since it implies that it also holds for all $\gamma < 2$. Next we proceed in several steps.

Step 1. Notations and rewriting of the problem. We consider $\sigma(t,x) := \theta(t)p(x)$, where $p:[0,1] \to \mathbb{R}$ and $\theta:(0,T) \to \mathbb{R}$ are two *smooth* functions satisfying the following properties (p and θ will be chosen later):

$$p(x) > 0 \quad \text{for all } x \in [0,1] \quad \text{and} \quad \theta(t) \to +\infty \quad \text{as } t \to 0^+, T^-. \tag{4.1}$$

For $R > 0$, we define

$$z(t,x) = e^{-R\sigma(t,x)} w(t,x), \tag{4.2}$$

where w solves (3.13) and (3.15). Notice that

$$z(0,\cdot) = z(T,\cdot) \equiv 0 \quad \text{in } (0,1), \tag{4.3}$$

and that z satisfies

$$\begin{cases} \left(e^{R\sigma}z\right)_t + \left(e^{R\sigma}z\right)_{xx} + \dfrac{\lambda}{x^2}e^{R\sigma}z + \dfrac{m}{x^\beta}e^{R\sigma}z = f, & (t,x) \in Q_T, \\ z(t,0) = 0, & t \in (0,T), \\ z(t,\cdot) = 0 \text{ in a neighbourhood of } x = 1, & t \in (0,T). \end{cases}$$

This equation may be recast as follows

$$P_R z = P_R^+ z + P_R^- z = f e^{-R\sigma}$$

where

$$P_R^+ z = R\sigma_t z + R^2\sigma_x^2 z + z_{xx} + \frac{\lambda}{x^2}z + \frac{m}{x^\beta}z, \qquad P_R^- z = z_t + R\sigma_{xx}z + 2R\sigma_x z_x.$$

Moreover, we have

$$\left\| f e^{-R\sigma} \right\|^2 = \left\| P_R^+ z \right\|^2 + \left\| P_R^- z \right\|^2 + 2\left\langle P_R^+ z, P_R^- z \right\rangle \geqslant 2\left\langle P_R^+ z, P_R^- z \right\rangle, \tag{4.4}$$

where $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively denote the usual norm and scalar product in $L^2(Q_T)$.

Step 2. Computation of the scalar product. In order to obtain a bound from below of the quantity $\|fe^{-R\sigma}\|^2$, we first compute the scalar product $\langle P_R^+ z, P_R^- z\rangle$:

Lemma 4.1. *The scalar product $\langle P_R^+ z, P_R^- z\rangle$ may be written as a sum of a distributed term* $\mathbf{A}$ *and a boundary term* $\mathbf{B}$:

$$\left\langle P_R^+ z, P_R^- z \right\rangle = \mathbf{A} + \mathbf{B},$$

where

$$\mathbf{A} = \iint_{Q_T} \left(-\frac{R}{2}\sigma_{tt} + \frac{R}{2}\sigma_{xxxx} - 2R^2\sigma_x\sigma_{xt} + 2\lambda R\frac{\sigma_x}{x^3} + \beta m R\frac{\sigma_x}{x^{\beta+1}} \right) z^2 - \iint_{Q_T} 2R^3\sigma_{xx}\sigma_x^2 z^2 - \iint_{Q_T} 2R\sigma_{xx}z_x^2$$

and

$$\mathbf{B} = \int_0^T \left[z_x z_t - \frac{R}{2}\sigma_{xxx}z^2 + \lambda R\frac{\sigma_x}{x^2}z^2 + mR\frac{\sigma_x}{x^\beta}z^2 + R^2\sigma_t\sigma_x z^2 + R^3\sigma_x^3 z^2 + R\sigma_{xx}zz_x + R\sigma_x z_x^2 \right]_0^1.$$

The proof of the above lemma is given later in Section 4.2. Next, using the relation $\sigma(t,x) = \theta(t)p(x)$ and the boundary conditions, we simplify the distributed and boundary terms (respectively denoted $\mathbf{A}$ and $\mathbf{B}$) as follows.

Lemma 4.2. *The distributed and boundary terms can be written as*

$$\mathbf{A} = -\frac{R}{2}\iint_{Q_T} \theta_{tt}pz^2 + \frac{R}{2}\iint_{Q_T} \theta p_{xxxx}z^2 - 2R^2\iint_{Q_T} \theta\theta_t p_x^2 z^2 + 2\lambda R\iint_{Q_T} \theta\frac{p_x}{x^3}z^2 + \beta m R\iint_{Q_T} \theta\frac{p_x}{x^{\beta+1}}z^2 - 2R^3\iint_{Q_T} \theta^3 p_{xx}p_x^2 z^2 - 2R\iint_{Q_T} \theta p_{xx}z_x^2,$$

$$\mathbf{B} = -\int_0^T \left(\lambda R\theta\frac{p_x}{x^2}z^2 + mR\theta\frac{p_x}{x^\beta}z^2 + R\theta p_x z_x^2 \right)_{x=0}.$$

The proof of this lemma is given later in Section 4.2.

Step 3. Choice of the weight functions θ and p. Let us now make precise the choice of the weight functions θ and p that we make here in order to treat the singularity.

Choice of θ. As stated in Theorem 3.2, we take

$$\forall t \in (0,T), \quad \theta(t) := \left(\frac{1}{t(T-t)} \right)^k \quad \text{with } k := 1 + \frac{2}{\gamma}.$$

This function satisfies:

$$\theta(t) \to +\infty \quad \text{as } t \to 0^+ \text{ and } t \to T^-,$$

and there is some $c > 0$ such that for all $t \in (0, T)$,

$$\left|\theta_t(t)\right| \leqslant c\theta(t)^{1+1/k} \quad \text{and} \quad \left|\theta_{tt}(t)\right| \leqslant c\theta(t)^{1+2/k}. \tag{4.5}$$

Choice of p. Now we define the function p:

$$\forall x \in [0,1], \quad p(x) := 1 - \frac{x^2}{2}.$$

It follows that p is smooth and positive on $[0, 1]$ and:

$$p_x = -x \quad \text{in } [0,1] \quad \text{hence} \quad p_x(0) = 0,$$
$$-p_{xx} = 1, \quad -p_{xx}p_x^2 = x^2 \quad \text{and} \quad p_{xxxx} = 0 \quad \text{in } [0,1].$$

Using the above choice of θ and p in Lemma 4.2, we deduce (see later in Section 4.2 for the proof of this lemma):

Lemma 4.3. *With this choice of θ and p, we have*

$$\begin{aligned}
\mathbf{A} = {} & 2R^3 \iint_{Q_T} \theta^3 x^2 z^2 + 2R \iint_{Q_T} \theta z_x^2 \\
& - \frac{R}{2} \iint_{Q_T} \theta_{tt} p z^2 - 2R^2 \iint_{Q_T} \theta\theta_t x^2 z^2 - 2\lambda R \iint_{Q_T} \theta \frac{z^2}{x^2} - \beta m R \iint_{Q_T} \theta \frac{z^2}{x^\beta}, \\
\mathbf{B} = {} & 0.
\end{aligned}$$

Step 4. Lower bound on the distributed term. We have the following lower bound on the term $\mathbf{A}$ (see later in Section 4.3 for the proof):

Lemma 4.4. *There exist some constants $R_0 > 0$ such that the distributed term $\mathbf{A}$ satisfies*:

$$\mathbf{A} \geqslant R^3 \iint_{Q_T} \theta^3 x^2 z^2 + 2R\left(\frac{1}{4} - \lambda\right) R \iint_{Q_T} \theta \frac{z^2}{x^2} + \frac{R}{2} \iint_{Q_T} \theta \frac{z^2}{x^\gamma}, \tag{4.6}$$

for all $R \geqslant R_0$.

Step 5. Conclusion. We deduce from Lemmas 4.1–4.4 that, for all $R \geqslant R_0$,

$$\begin{aligned}
& R^3 \iint_{Q_T} \theta^3 x^2 z^2 + 2R\left(\frac{1}{4} - \lambda\right) R \iint_{Q_T} \theta \frac{z^2}{x^2} + \frac{R}{2} \iint_{Q_T} \theta \frac{z^2}{x^\gamma} \\
& \quad \leqslant \mathbf{A} + \mathbf{B} = \langle P_R^+ z, P_R^- z \rangle \leqslant \frac{1}{2} \| f e^{-R\sigma} \|^2 = \frac{1}{2} \iint_{Q_T} f^2 e^{-2R\sigma}.
\end{aligned}$$

Since $z = we^{-R\sigma}$, we deduce that, for all $R \geqslant R_0$:

$$R^3 \iint_{Q_T} \theta^3 x^2 w^2 e^{-2R\sigma} + 2R\left(\frac{1}{4} - \lambda\right) R \iint_{Q_T} \theta \frac{w^2}{x^2} e^{-2R\sigma} + \frac{R}{2} \iint_{Q_T} \theta \frac{w^2}{x^\gamma} e^{-2R\sigma}$$
$$\leqslant \frac{1}{2} \iint_{Q_T} f^2 e^{-2R\sigma},$$

which ends the proof of Theorem 3.2.

4.2. *Proof of Lemmas 4.1–4.3 (expression of the scalar product)*

Proof of Lemma 4.1. Let us write

$$\left\langle P_R^+ z, P_R^- z\right\rangle = Q_1 + Q_2 + Q_3 + Q_4 + Q_5,$$

where

$$Q_1 := \left\langle R\sigma_t z + R^2 \sigma_x^2 z + z_{xx}, z_t\right\rangle,$$

$$Q_2 := R^2 \langle \sigma_t z, \sigma_{xx} z + 2\sigma_x z_x\rangle,$$

$$Q_3 := R^3 \left\langle \sigma_x^2 z, \sigma_{xx} z + 2\sigma_x z_x\right\rangle,$$

$$Q_4 := R\langle z_{xx}, \sigma_{xx} z + 2\sigma_x z_x\rangle,$$

$$Q_5 := \left\langle \lambda \frac{1}{x^2} z + m \frac{1}{x^\beta} z, z_t + R\sigma_{xx} z + 2R\sigma_x z_x\right\rangle.$$

First, we compute Q_1. Integrating by parts, we get

$$Q_1 = \iint_{Q_T} \left(R\sigma_t z + R^2 \sigma_x^2 z + z_{xx}\right) z_t = \iint_{Q_T} \left(R\sigma_t + R^2 \sigma_x^2\right) \left(\frac{z^2}{2}\right)_t + \iint_{Q_T} z_{xx} z_t$$

$$= \left[\int_0^1 \frac{1}{2}\left(R\sigma_t + R^2 \sigma_x^2\right) z^2\right]_0^T - \iint_{Q_T} \frac{1}{2}\left(R\sigma_t + R^2 \sigma_x^2\right)_t z^2 + \int_0^T [z_x z_t]_0^1 - \iint_{Q_T} z_x z_{xt}$$

$$= \left[\int_0^1 \left(R\sigma_t + R^2 \sigma_x^2\right) \frac{1}{2} z^2 - \frac{1}{2} z_x^2\right]_0^T - \iint_{Q_T} \frac{1}{2}\left(R\sigma_t + R^2 \sigma_x^2\right)_t z^2 + \int_0^T [z_x z_t]_0^1.$$

By (4.3), the time integrals vanish. Hence,

$$Q_1 = \int_0^T [z_x z_t]_0^1 + \iint_{Q_T} \left(-\frac{1}{2} R\sigma_{tt} - R^2 \sigma_x \sigma_{xt}\right) z^2. \tag{4.7}$$

The term Q_2 becomes:

$$Q_2 = R^2 \iint_{Q_T} \sigma_t z(\sigma_{xx} z + 2\sigma_x z_x) = R^2 \iint_{Q_T} \sigma_t \sigma_{xx} z^2 + \sigma_t \sigma_x \left(z^2\right)_x$$

$$= R^2 \iint_{Q_T} \sigma_t \sigma_{xx} z^2 + R^2 \int_0^T \left[\sigma_t \sigma_x z^2\right]_0^1 - R^2 \iint_{Q_T} (\sigma_t \sigma_x)_x z^2.$$

Therefore,

$$Q_2 = R^2 \int_0^T \left[\sigma_t \sigma_x z^2\right]_0^1 - R^2 \iint_{Q_T} \sigma_x \sigma_{xt} z^2. \tag{4.8}$$

The term Q_3 can be simplified as follows:

$$Q_3 = R^3 \iint_{Q_T} \sigma_x^2 z(\sigma_{xx} z + 2\sigma_x z_x) = R^3 \iint_{Q_T} \sigma_{xx} \sigma_x^2 z^2 + R^3 \iint_{Q_T} \sigma_x^3 \left(z^2\right)_x$$

$$= R^3 \iint_{Q_T} \sigma_{xx} \sigma_x^2 z^2 + R^3 \int_0^T \left[\sigma_x^3 z^2\right]_0^1 - R^3 \iint_{Q_T} \left(\sigma_x^3\right)_x z^2.$$

Thus,

$$Q_3 = R^3 \int_0^T \left[\sigma_x^3 z^2\right]_0^1 - 2R^3 \iint_{Q_T} \sigma_{xx} \sigma_x^2 z^2. \tag{4.9}$$

Next we compute Q_4:

$$Q_4 = R \iint_{Q_T} z_{xx}(\sigma_{xx} z + 2\sigma_x z_x) = R \int_0^T [z_x \sigma_{xx} z]_0^1 - R \iint_{Q_T} z_x (\sigma_{xx} z)_x + R \iint_{Q_T} \sigma_x \left(z_x^2\right)_x$$

$$= R \int_0^T [\sigma_{xx} z z_x]_0^1 - R \iint_{Q_T} \sigma_{xx} z_x^2 + \sigma_{xxx} z z_x + R \int_0^T \left[\sigma_x z_x^2\right]_0^1 - R \iint_{Q_T} \sigma_{xx} z_x^2.$$

Hence

$$Q_4 = R\int_0^T \left[\sigma_{xx} z z_x + \sigma_x z_x^2\right]_0^1 - 2R\iint_{Q_T} \sigma_{xx} z_x^2 - R\iint_{Q_T} \sigma_{xxx} z z_x.$$

Since

$$-\iint_{Q_T} R\sigma_{xxx} z z_x = -\iint_{Q_T} \frac{R}{2}\sigma_{xxx}\left(z^2\right)_x = -\int_0^T \left[\frac{R}{2}\sigma_{xxx} z^2\right]_0^1 + \iint_{Q_T} \frac{R}{2}\sigma_{xxxx} z^2,$$

we obtain

$$Q_4 = R\int_0^T \left[\sigma_{xx} z z_x + \sigma_x z_x^2 - \frac{1}{2}\sigma_{xxx} z^2\right]_0^1 - 2R\iint_{Q_T} \sigma_{xx} z_x^2 + \frac{R}{2}\iint_{Q_T} \sigma_{xxxx} z^2. \tag{4.10}$$

Finally, it remains to compute Q_5. Using (4.3), we obtain:

$$\begin{aligned}
Q_5 &= \lambda\iint_{Q_T} \frac{1}{x^2}\left(\frac{z^2}{2}\right)_t + R\frac{\sigma_{xx}}{x^2} z^2 + R\frac{\sigma_x}{x^2}\left(z^2\right)_x + m\iint_{Q_T} \frac{1}{x^\beta}\left(\frac{z^2}{2}\right)_t + R\frac{\sigma_{xx}}{x^\beta} z^2 + R\frac{\sigma_x}{x^\beta}\left(z^2\right)_x \\
&= \lambda R\iint_{Q_T} \frac{\sigma_{xx}}{x^2} z^2 + \lambda R\int_0^T \left[\frac{\sigma_x}{x^2} z^2\right]_0^1 - \lambda R\iint_{Q_T} \left(\frac{\sigma_x}{x^2}\right)_x z^2 + mR\iint_{Q_T} \frac{\sigma_{xx}}{x^\beta} z^2 \\
&\quad + mR\int_0^T \left[\frac{\sigma_x}{x^\beta} z^2\right]_0^1 - mR\iint_{Q_T} \left(\frac{\sigma_x}{x^\beta}\right)_x z^2.
\end{aligned}$$

Hence

$$Q_5 = \lambda R\int_0^T \left[\frac{\sigma_x}{x^2} z^2\right]_0^1 + mR\int_0^T \left[\frac{\sigma_x}{x^\beta} z^2\right]_0^1 + 2\lambda R\iint_{Q_T} \frac{\sigma_x}{x^3} z^2 + \beta mR\iint_{Q_T} \frac{\sigma_x}{x^{\beta+1}} z^2. \tag{4.11}$$

Putting (4.7)–(4.11) together, one obtains Lemma 4.1. □

Proof of Lemma 4.2. Using $\sigma(t,x) = \theta(t)p(x)$, the computation of $\mathbf{A}$ directly follows from Lemma 4.1. On the other hand, we obtain for $\mathbf{B}$:

$$\mathbf{B} = \int_0^T \Bigg[z_x z_t - \frac{R}{2}\theta p_{xxx} z^2 + \lambda R\theta\frac{p_x}{x^2} z^2 + mR\theta\frac{p_x}{x^\beta} z^2 + R^2\theta\theta_t p p_x z^2$$

$$+ R^3\theta^3 p_x^3 z^2 + R\theta p_{xx} z z_x + R\theta p_x z_x^2 \Big]_0^1$$

$$= -\int_0^T \left(z_x z_t - \frac{R}{2}\theta p_{xxx} z^2 + \lambda R\theta \frac{p_x}{x^2} z^2 + mR\theta \frac{p_x}{x^\beta} z^2 + R^2\theta\theta_t p p_x z^2 \right.$$
$$\left. + R^3\theta^3 p_x^3 z^2 + R\theta p_{xx} z z_x + R\theta p_x z_x^2 \right)\Big|_{x=0}$$

since $z(t,\cdot) \equiv 0$ in a neighbourhood of $x = 1$. Next we use the fact that $z(t, 0) = 0$ and that p is smooth in $[0, 1]$ to obtain

$$\mathbf{B} = -\int_0^T \left(\lambda R\theta \frac{p_x}{x^2} z^2 + mR\theta \frac{p_x}{x^\beta} z^2 + R\theta p_x z_x^2 \right)\Big|_{x=0}. \qquad \square$$

Proof of Lemma 4.3. The expression of $\mathbf{A}$ directly follows from the choice of p. On the other hand, $\mathbf{B}$ becomes

$$\mathbf{B} = \int_0^T \left(\lambda R\theta \frac{z^2}{x} + mR\theta \frac{z^2}{x^{\beta-1}} + R\theta x z_x^2 \right)\Big|_{x=0} = \int_0^T \left(\lambda R\theta \frac{z^2}{x} + mR\theta \frac{z^2}{x^{\beta-1}} \right)\Big|_{x=0}.$$

Thanks to the following lemma, it implies that $\mathbf{B} = 0$:

Lemma 4.5.

$$\forall z \in H_0^1(0,1), \qquad \frac{z(x)^2}{x} \to 0 \quad \text{as } x \to 0^+.$$

This ends the proof of Lemma 4.3. It remains now to prove Lemma 4.5. $\square$

Proof of Lemma 4.5. Let z be given in $H_0^1(0, 1)$. For all $x \in [0, 1]$, we can write

$$\big|z(x)\big| = \left| \int_0^x z_x(s)\, ds \right| \leqslant \left(\int_0^x z_x(s)^2\, ds \right)^{1/2} \sqrt{x}.$$

Hence

$$\frac{|z(x)|^2}{|x|} \leqslant \int_0^x z_x(s)^2\, ds \to 0 \quad \text{as } x \to 0^+,$$

since z_x belongs to $L^2(0, 1)$. $\square$

4.3. Proof of Lemma 4.4 (lower bound on the distributed terms)

We now want to get lower bounds for the distributed terms appearing in the scalar product $\langle P_R^+ z, P_R^- z\rangle$. By Lemma 4.3, we have $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4$, where

$$\mathbf{A}_1 := 2R^3 \iint_{Q_T} \theta^3 x^2 z^2 + 2R \iint_{Q_T} \theta z_x^2, \qquad \mathbf{A}_2 := -\frac{R}{2} \iint_{Q_T} \theta_{tt} p z^2,$$

and

$$\mathbf{A}_3 := -2R^2 \iint_{Q_T} \theta\theta_t x^2 z^2, \qquad \mathbf{A}_4 := -2\lambda R \iint_{Q_T} \theta \frac{z^2}{x^2} - \beta m R \iint_{Q_T} \theta \frac{z^2}{x^\beta}.$$

Let us first estimate the term $\mathbf{A}_3$. By (4.5), we know that $|\theta_t| \leqslant c\theta^{1+1/k} \leqslant c\theta^2$ since $k > 1$, hence $|\theta\theta_t| \leqslant c\theta^3$, and we obtain

$$|\mathbf{A}_3| \leqslant 2R^2 \iint_{Q_T} |\theta\theta_t| x^2 z^2 \leqslant cR^2 \iint_{Q_T} \theta^3 x^2 z^2. \tag{4.12}$$

Hence

$$\mathbf{A} \geqslant \left(2R^3 - cR^2\right) \iint_{Q_T} \theta^3 x^2 z^2 + 2R \iint_{Q_T} \theta z_x^2 + \mathbf{A}_2 + \mathbf{A}_4. \tag{4.13}$$

In the following, we produce estimates of the last two terms $\mathbf{A}_2$ and $\mathbf{A}_4$.

More precisely, we will use the following improved form of (3.14) (see [27, Section 2.1.6]):

Lemma 4.6. *For all $n > 0$ and $0 < \gamma < 2$, there exists some positive constant $C_0 = C_0(\gamma, n) > 0$ such that*

$$\forall z \in H_0^1(0,1), \quad \int_0^1 z_x^2\, dx + C_0 \int_0^1 z^2\, dx \geqslant \frac{1}{4} \int_0^1 \frac{z^2}{x^2}\, dx + n \int_0^1 \frac{z^2}{x^\gamma}\, dx. \tag{4.14}$$

Let us now continue the proof of Lemma 4.4. Since $0 \leqslant \beta < \gamma < 2$, we have

$$\begin{aligned}
\mathbf{A}_4 &= -2\lambda R \iint_{Q_T} \theta \frac{z^2}{x^2} - \beta m R \iint_{Q_T} \theta \frac{z^2}{x^\beta} \geqslant -2\lambda R \iint_{Q_T} \theta \frac{z^2}{x^2} - \beta |m| R \iint_{Q_T} \theta \frac{z^2}{x^\gamma} \\
&= 2R\left(\frac{1}{4} - \lambda\right) \iint_{Q_T} \theta \frac{z^2}{x^2} + R \iint_{Q_T} \theta \frac{z^2}{x^\gamma} - 2R\left[\frac{1}{4} \iint_{Q_T} \theta \frac{z^2}{x^2} + \left(\frac{\beta|m|}{2} + \frac{1}{2}\right) \iint_{Q_T} \theta \frac{z^2}{x^\gamma}\right].
\end{aligned}$$

On the other hand, for $C > 0$ large enough we have, as a consequence of Lemma 4.6,

$$\left|\frac{1}{4}\iint_{Q_T} \theta \frac{z^2}{x^2} + \left(\frac{\beta|m|}{2} + \frac{1}{2}\right)\iint_{Q_T} \theta \frac{z^2}{x^\gamma}\right| \leqslant \iint_{Q_T} \theta z_x^2 + C\iint_{Q_T} \theta z^2.$$

Hence

$$\mathbf{A}_4 \geqslant 2R\left(\frac{1}{4} - \lambda\right)\iint_{Q_T} \theta \frac{z^2}{x^2} + R\iint_{Q_T} \theta \frac{z^2}{x^\gamma} - 2R\iint_{Q_T} \theta z_x^2 - 2RC\iint_{Q_T} \theta z^2.$$

From (4.13), it follows that

$$\begin{aligned} \mathbf{A} \geqslant {}& \left(2R^3 - cR^2\right)\iint_{Q_T} \theta^3 x^2 z^2 + 2R\left(\frac{1}{4} - \lambda\right)\iint_{Q_T} \theta \frac{z^2}{x^2} + R\iint_{Q_T} \theta \frac{z^2}{x^\gamma} \\ & + \mathbf{A}_2 - 2RC\iint_{Q_T} \theta z^2. \end{aligned} \tag{4.15}$$

Next, we need to estimate the term $\mathbf{A}_2$ and the last term in the above inequality. By (4.5), we have

$$|\theta_{tt}|\|p\|_\infty \leqslant C\theta^{1+2/k}$$

for some $C > 0$. It follows that,

$$\left|\mathbf{A}_2 - 2RC\iint_{Q_T} \theta z^2\right| \leqslant \frac{R}{2}\iint_{Q_T} |\theta_{tt}| p z^2 + 2RC\iint_{Q_T} \theta z^2 \leqslant CR\iint_{Q_T} \theta^{1+2/k} z^2.$$

We set

$$q = \frac{2+\gamma}{\gamma}, \qquad q' = \frac{2+\gamma}{2},$$

so that

$$q^{-1} + (q')^{-1} = 1.$$

Then, for all $\varepsilon > 0$, we have

$$\begin{aligned} \iint_{Q_T} \theta^{1+2/k} z^2 &= \iint_{Q_T} \left(\frac{1}{\varepsilon}\theta^{1+2/k-1/q'} x^{\gamma/q'} z^{2/q}\right)\left(\varepsilon\theta^{1/q'} x^{-\gamma/q'} z^{2/q'}\right) \\ &\leqslant \frac{C}{\varepsilon^q}\iint_{Q_T} \theta^{q(1+2/k-1/q')} x^{\gamma q/q'} z^2 + \varepsilon^{q'} C\iint_{Q_T} \theta \frac{z^2}{x^\gamma}. \end{aligned}$$

Note that

$$q\left(1+\frac{2}{k}-\frac{1}{q'}\right)=3, \qquad \gamma q/q'=2.$$

Thus,

$$\left|\mathbf{A}_2-2RC\iint\limits_{Q_T}\theta z^2\right|\leqslant\frac{CR}{\varepsilon^q}\iint\limits_{Q_T}\theta^3x^2z^2+\varepsilon^{q'}CR\iint\limits_{Q_T}\theta\frac{z^2}{x^\gamma}.$$

Putting this estimate in (4.15) we obtain:

$$\mathbf{A}\geqslant\left(2R^3-cR^2-\frac{CR}{\varepsilon^q}\right)\iint\limits_{Q_T}\theta^3x^2z^2+2R\left(\frac{1}{4}-\lambda\right)\iint\limits_{Q_T}\theta\frac{z^2}{x^2}$$
$$+R\big(1-\varepsilon^{q'}C\big)\iint\limits_{Q_T}\theta\frac{z^2}{x^\gamma}.$$

Taking $\varepsilon>0$ small enough and next R large enough, we get the inequality in Lemma 4.4.

5. The supercritical case

In this section, we consider the supercritical case $\mu>\mu^\star(N)$. We first give the precise definition of H_μ. Next we prove point (ii) of Theorem 2.2.

5.1. *Definition of H_μ and well-posedness in H_μ*

We recall that, as proved in [1], in the supercritical case, positive solutions blow up instantaneously. This holds, in fact, at the level of radially symmetric solutions and it is simply due to the singularity of solutions at the origin, which excludes even the interpretation of the equation in the sense of distributions.

However, as we have seen, using a spherical harmonics decomposition one can see that, for solutions that oscillate sufficiently fast on the unit sphere, the effect of the singular potential can be compensated. This allows establishing well-posedness for a subclass of initial data.

Assume that $\mu>\mu^\star(N)$ and let $k_\mu\in\mathbb{N}$ be defined by

$$k_\mu:=\min\big\{k\in\mathbb{N}\ \big|\ \lambda_k\leqslant 1/4\big\},$$

where λ_k is defined by (3.11). Then we consider the space

$$H_\mu:=\left\{z\in L^2(\Omega)\ \middle|\ z(r\sigma)=\sum_{k=k_\mu}^{+\infty}\sum_{l=1}^{l_k}z^{k,l}(r)f^{k,l}(\sigma)\text{ with}\right.$$
$$\left.\sum_{k=k_\mu}^{+\infty}\sum_{l=1}^{l_k}\int_0^1 z^{k,l}(r)^2r^{N-1}\,dr<+\infty\right\}.$$

Following the analysis of [33], one can prove that, even in the case $\mu > \mu^\star(N)$, the well-posedness result of Theorem 2.1 is still true provided $u_0 \in H_\mu$.

5.2. Null controllability within the class of initial conditions in H_μ

Moreover, using the approach developed in Section 3, one can also easily see that such u_0 belonging to H_μ are also null-controllable. More precisely, we can prove

Theorem 5.1. *Let Ω be a ball of $\mathbb{R}^N$. Assume the control subset ω fulfills the geometric condition* (2.1) *and $\mu > \mu^\star(N)$. Then, for all $u_0 \in H_\mu$, with H_μ as above, there exists $h \in L^2(Q_T)$ such that the solution of* (1.2) *satisfies $u(T, X) \equiv 0$ for a.e. $X \in \Omega$.*

Let us briefly describe the ideas of the proof of Theorem 5.1.

Step 1. In order to get Theorem 5.1, we first prove that, under the assumption $\mu > \mu^\star(N)$, the observability inequality (2.3) still holds within the class of v_T belonging to H_μ.

This can easily be done since (3.5) may still be transformed into the family of problems (3.10) indexed by k. But, thanks to the condition $v_T \in H_\mu$, the index k is now such that $k \geqslant k_\mu$. Hence the parameter λ_k that appears in (3.10) still satisfies the condition $\lambda_k \leqslant 1/4$. The rest of the proof is identical to the proof made in Section 3.

Step 2. Now it remains to deduce the controllability result stated in Theorem 5.1 from this partial observability inequality.

As in the subcritical case, once (2.3) is known to hold for the solutions of the adjoint system with initial data (at time $t = T$) in H_μ, the control h can be taken as being $h = v^\star$ in $(0, T) \times \omega$ where $v^\star$ is the solution of (2.2) with the initial data $v_T^\star$ minimizing the functional (2.4) within the subspace $\mathcal{H} \cap H_\mu$ constituted by the initial data $v_T \in H_\mu$ such that the corresponding solution of (2.2) is such that

$$\|v_T\|_{\mathcal{H}} = \left[\int_0^T \int_\omega v^2 \, dX \, dt \right]^{1/2} < +\infty.$$

The observability inequality (2.3) within H_μ guarantees that the functional $J : \mathcal{H} \cap H_\mu \to \mathbb{R}$, in addition to being continuous and convex, is coercive. This guarantees the existence of the minimizer $v_T^\star$, which is in fact unique by strict convexity.

Finally the fact that $DJ(v_T^\star) = 0$ implies the null controllability condition (1.3). In principle this only implies $\pi_\mu u(T) = 0$, π_μ being the projection over the spherical harmonics components involved in H_μ. But the fact that the initial data to be controlled and the control lie in H_μ, together with the fact that the various spherical harmonics components do not interact, allows seeing that, actually, the whole solution $u(T)$ vanishes.

6. Comments and open problems

In this last section, we present some further results and discuss some possible extensions and open questions.

6.1. Choice of the weight functions in the 1-d Carleman estimates

Let us comment on the form of the Carleman estimates given in Theorem 3.2. In comparison with the standard Carleman estimates, the weight function θ is only slightly modified. On the contrary, the choice of the weight function p is not standard. Indeed it is carefully chosen to treat the singularity at $x = 0$. Moreover, some other weights such as x^2 also appear in the formulation of the inequality that we obtain.

Our proof is inspired by the method introduced in [8,26] to prove null controllability for parabolic equations with moderate degenerate coefficients combining Hardy inequalities and this kind of Carleman inequalities with adapted weights.

In the present case, we face the added difficulty that the estimates need to be uniform with respect to the parameter $\lambda \leqslant 1/4$. This requires focusing on solutions which also satisfy (3.15). Without this last condition the observability constants would depend strongly on λ and blow up when $\lambda \to -\infty$.

6.2. Extension to more general potentials

The result of this paper may also be extended to more general singular potentials

$$V(X) := \frac{\mu}{|X|^2} + \frac{m}{|X|^\beta}, \tag{6.1}$$

for $m \in \mathbb{R}$ and $0 \leqslant \beta < 2$. One can prove the following result.

Theorem 6.1 *(Controllability). Assume the control subset ω fulfills the geometric condition* (2.1) *and let $V(X)$ be defined by* (6.1) *with $m \in \mathbb{R}$ and $0 \leqslant \beta < 2$. Assume $\mu \leqslant \mu^\star(N)$. Then, for all $u_0 \in L^2(\Omega)$, there exists $h \in L^2(Q_T)$ such that the solution of* (1.2) *satisfies $u(T, X) \equiv 0$ for a.e. $X \in \Omega$.*

Following the procedure developed in Section 3, we easily see that the proof of Theorem 6.1 exactly follows from the uniform Carleman estimates given in Theorem 3.2.

6.3. Null controllability in 1-d

It is easy to see that Theorem 3.2 also implies a null controllability result in 1-d that is similar to the result given in Theorem 2.2. For the sake of completeness we state it below:

Theorem 6.2. *Assume that $\lambda \leqslant 1/4$, $m \in R$ and $0 \leqslant \beta < 2$, and consider $T > 0$ and $0 \leqslant x_1 < x_2 \leqslant 1$. Then, for all $u_0 \in L^2(0, 1)$, there exists $h \in L^2(Q_T)$ such that the solution of*

$$\begin{cases} u_t - u_{xx} - \dfrac{\lambda}{x^2}u - \dfrac{m}{x^\beta}u = h\chi_{(x_1,x_2)}, & (t, x) \in Q_T, \\ u(t, 0) = 0 = u(t, 1), & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, 1), \end{cases} \tag{6.2}$$

satisfies $u(T, x) = 0$ for a.e. $x \in (0, 1)$.

6.4. Carleman estimates in N-d

As mentioned in Remark 3.4, the 1-d Carleman estimates stated in Theorem 3.2 also provide a similar result in the N-d spatial domain $\mathbb{B}^N$. Note that, so far, we have only used these Carleman inequalities to obtain observability estimates and null controllability results. Here we use them to show the kind of Carleman inequalities that hold in the multi-dimensional case. These inequalities have slight differences with those that hold for bounded potentials, as we shall see.

Let us consider the following problem

$$\begin{cases} w_t + \Delta w + \dfrac{\mu}{|X|^2} w = g, & (t, X) \in (0, T) \times \mathbb{B}^N, \\ w(t, X) = 0, & (t, X) \in (0, T) \times \partial\mathbb{B}^N, \\ w(T, X) = w_T(X), & X \in \mathbb{B}^N, \end{cases} \tag{6.3}$$

where $g \in L^2((0, T) \times \mathbb{B}^N)$ and $w_T \in L^2(\mathbb{B}^N)$.

As in Theorem 3.2, our result concerns the solutions of (6.3) that vanish in a neighbourhood of the boundary $\partial\mathbb{B}^N$ of the domain $\mathbb{B}^N$. More precisely, we consider the solutions of (6.3) satisfying

$$w(t, X) = 0 \quad \text{for all } t \in (0, T) \quad \text{and} \quad X \text{ such that } 1 - \eta < |X| < 1, \tag{6.4}$$

for some $0 < \eta < 1$. Then the following Carleman inequality holds for these solutions of (6.3).

Theorem 6.3 *(Carleman estimates). Assume that $\mu \leqslant \mu^\star(N)$ and, for every $\gamma < 2$, consider the function σ defined in Theorem* 3.2. *Then, there exists $R_0 > 0$ such that, for all $R \geqslant R_0$, the solutions w of* (6.3) *such that* (6.4) *holds satisfy*

$$R^3 \int_0^T \int_{\mathbb{B}^N} \theta^3(t)|X|^2 w(t, X)^2 e^{-2R\sigma(t,|X|)}\, dX\, dt + \frac{R}{2} \int_0^T \int_{\mathbb{B}^N} \theta(t) \frac{w(t, X)^2}{|X|^\gamma} e^{-2R\sigma(t,|X|)}\, dX\, dt$$
$$\leqslant \frac{1}{2} \int_0^T \int_{\mathbb{B}^N} f(t, X)^2 e^{-2R\sigma(t,|X|)}\, dX\, dt. \tag{6.5}$$

The proof of Theorem 6.3 is a direct consequence of Theorem 3.2 and of the procedure based on a decomposition in spherical harmonics described in Section 3.3.

Note that this Carleman inequality is different from the classical one in Theorem 3.1 for bounded potentials. We see in particular that in the left-hand side term of this new inequality there is a degenerate density function $|X|^2$. The exponential factor of the weight function is also different.

Theorem 6.3 also provides global Carleman estimates for the solutions of (6.3) in a general domain Ω with an observation region ω satisfying condition (2.1). Indeed, with a cut-off argument such as in Step 2 of Section 3.1, Theorem 6.3 associated to standard Carleman estimates allows to estimate the solutions w of (6.3) by f and by the values of w over the observation region $(0, T) \times \omega$. (Here the weight functions need to be piecewisely defined in order to be equal to the weight functions of standard Carleman estimates in the exterior domain to ω and to be equal to the weight functions of the new singular Carleman estimates in the interior domain to ω.)

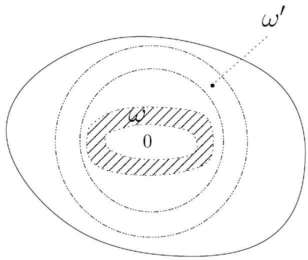

Fig. 6.1.

6.5. Geometric assumption on the control region

Our method, based on a decomposition in spherical harmonics, strongly uses the fact that the control domain ω contains some annular set centered around the singularity. The case of a general geometry for ω cannot be treated in the same way.

Jérôme Le Rousseau [25] observed however that our arguments also work for a more general class of subdomains ω. It is not necessary to assume that ω contains an annular set centered around the singularity. It suffices that ω surrounds the singularity and that the exterior part of ω contains some annular set centered around the singularity.

Indeed, in that case, let us denote D_1 the domain to the exterior of ω and D_2 the domain to the interior of ω. Then we have $\Omega = D_1 \cup \omega \cup D_2$ with $0 \in D_2$ and with D_1 containing some annular set ω' centered around the singularity:

$$\omega' := \left\{ X \in \mathbb{R}^N \;\middle|\; r_1 < |X| < r_2 \right\} \subset D_1,$$

for some constants r_1, r_2 such that $0 \leqslant r_1 < r_2$, see Fig. 6.1. To fix ideas, we may also assume that $r_2 < 1$.

As previously, the proof of Theorem 2.2 reduces to the two partial observability inequalities (3.2) and (3.3). In the exterior domain D_1, we use the same cut-off argument and apply standard Carleman estimates to obtain (3.2). Next, using another cut-off argument, we apply our new Carleman estimates in $\mathbb{B}^N$ to get

$$\int_{T/4}^{3T/4} \int_{D_2} v(t,X)^2 \, dX \, dt \leqslant C \int_0^T \int_{\omega'} v(t,X)^2 \, dX \, dt. \tag{6.6}$$

Since $\omega' \subset D_1$, we deduce from (6.6) associated to (3.2) that (3.3) also holds.

Finally, as mentioned in the introduction, the case of an arbitrary nonempty open subset ω of $\Omega \setminus \{0\}$ has recently been solved by Ervedoza [14] using weighted Carleman inequalities similar to (6.5). However, in [14], these inequalities are derived globally working in the whole domain and with the complete solution as originally done for the standard equation in [17].

6.6. Open questions in the supercritical case

In the supercritical case $\mu > \mu^\star(N)$, two questions arise.

As mentioned in the introduction, the question of whether, for general initial conditions u_0 and control h of indefinite sign, the solutions may exists and be controllable or still blows up instantaneously (as in the case of positive solutions without control) whatever h is constitutes the first open question.

The second open question is whether one can generalize the partial result of controllability given in Section 5 to the case of a general domain Ω. Indeed we have seen that, in the case where Ω is a ball, in the supercritical case one can identify an invariant subspace of oscillating solutions for which the problem is well-posed and null-controllable. The case of a general domain Ω is much more delicate since all the components of a spherical harmonics decomposition interact. Thus, identifying a subspace in which the problem is well-posed and null-controllable for general domains is an open problem.

6.7. Semi-linear equations

This paper has been devoted to the linear problem. Of course it would be natural to address the issue of controlling nonlinear versions of these equations. One of the very first ones to be considered would be:

$$\begin{cases} u_t - \Delta u - \lambda e^u = h\chi_\omega, & (t, X) \in (0, T) \times \Omega, \\ u(t, X) = 0, & (t, X) \in (0, T) \times \Gamma, \\ u(0, X) = u_0(X), & X \in \Omega, \end{cases} \tag{6.7}$$

with $u_0 \in L^2(\Omega)$.

One of the most natural problems to be addressed is the control to a stationary state, i.e. driving the solution of (6.7) to a stationary solution u_s in time $t = T$ by means of a suitable control action h localized in ω.

There is a rich literature on the structure of the set of equilibrium solutions for this problem, i.e. on the solutions of the semi-linear elliptic equation (see [28], [5] and the references therein, for instance):

$$-\Delta u - \lambda e^u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma. \tag{6.8}$$

For some particular values of λ there are bounded solutions. In that case the local controllability to them can be proved easily by the existing fixed-point methods (see, for instance, [16]) since one can work within the frame of bounded solutions, without using the theory developed in this article to deal with singular potentials.

However, as mentioned in the introduction, some other stationary solutions are singular. It is the case, for instance, when $\lambda = 2(N-2)$ for which there exists a singular stationary solution: $u_s(x) = -2\log(|x|)$. After "linearization" of (6.7) around this equilibrium we obtain the following linear system with singular potential:

$$v_t - \Delta v - \frac{\lambda}{|x|^2} v = h\chi_\omega.$$

Note that this particular value of λ satisfies the bound $\lambda \leqslant \mu^*(N)$ if and only if $N \geqslant 10$. Thus, the null controllability results of this paper apply for large dimensions N.

Note however that it is hard to justify rigorously the linearization process. In fact, it is well known that linearization fails even in the elliptic context since, in particular, no stationary solutions exist for $\lambda > 2(N-2)$. This is due, in particular, to the fact that, even if the operator $-\Delta - \frac{\lambda}{|x|^2} I$ defines an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ (or from H to H' in the critical case), closedness of functions in $H_0^1(\Omega)$ does not imply closedness of their exponentials in $H^{-1}(\Omega)$.

Therefore, the existing techniques for proving the local controllability of the nonlinear problem to a singular stationary solution do not apply. This is an interesting open subject of research.

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